

ORIENTED PERCOLATION IN A RANDOM ENVIRONMENT

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ABSTRACT. On the lattice $\tilde{\mathbb{Z}}_+^2 := \{(x, y) \in \mathbb{Z} \times \mathbb{Z}_+ : x + y \text{ is even}\}$ we consider the following oriented (northwest-northeast) site percolation: the lines $H_i := \{(x, y) \in \tilde{\mathbb{Z}}_+^2 : y = i\}$ are first declared to be *bad* or *good* with probabilities δ and $1 - \delta$ respectively, independently of each other. Given the configuration of lines, sites on good lines are open with probability $p_G > p_c$, the critical probability for the standard oriented site percolation on $\mathbb{Z}_+ \times \mathbb{Z}_+$, and sites on bad lines are open with probability p_B , some small positive number, independently of each other. We show that given any pair $p_G > p_c$ and $p_B > 0$, there exists a $\delta(p_G, p_B) > 0$ small enough, so that for $\delta \leq \delta(p_G, p_B)$ there is a strictly positive probability of oriented percolation to infinity from the origin.

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1. INTRODUCTION

On the lattice $\tilde{\mathbb{Z}}_+^2 := \{(x, y) \in \mathbb{Z} \times \mathbb{Z}_+ : x + y \text{ is even}\}$ we consider the following oriented (northwest-northeast) site percolation: the lines $H_i := \{(x, y) \in \tilde{\mathbb{Z}}_+^2 : y = i\}$ are first declared to be *bad* or *good* with probabilities δ and $1 - \delta$ respectively, independently of each other. Given the configuration of lines, sites on good lines are open with probability p_G , and sites on bad lines are open with probability p_B , independently of each other. More formally, on a suitable probability space (Ω, \mathcal{A}, P) , we consider a Bernoulli sequence $\xi = (\xi_i : i \in \mathbb{Z}_+)$ with $P(\xi_i = 1) = \delta = 1 - P(\xi_i = 0)$, which determines H_i to be bad or good, and a family of occupation variables $(\eta_z : z \in \tilde{\mathbb{Z}}_+^2)$ which are conditionally independent given ξ ,

with $P(\eta_z = 1 \mid \xi) = p_B = 1 - P(\eta_z = 0 \mid \xi)$ if $z \in H_i$ with $\xi_i = 1$, and $P(\eta_z = 1 \mid \xi) = p_G = 1 - P(\eta_z = 0 \mid \xi)$ if $z \in H_i$ with $\xi_i = 0$. If $\eta_z = 1$ the site z is *open*, and otherwise it is *closed*. An *open oriented path* on $\tilde{\mathbb{Z}}^2$ is a path along which the second coordinate is strictly increasing and all of whose vertices are open. The *open cluster* of a vertex $z \in \tilde{\mathbb{Z}}^2$ is the collection of sites which can be reached from z by an open oriented path. This cluster is denoted by C_z and the open cluster of the origin is denoted by C_0 . We always include z itself in C_z , whether z is open or not. We say that *percolation occurs* if C_0 is infinite with positive probability. This description of percolation is of course obtained by rotating the standard picture on $\mathbb{Z}_+ \times \mathbb{Z}_+$ by $\pi/4$ counterclockwise.

The interesting situation is when $p_G > p_c$, the critical probability for the standard oriented site percolation on $\mathbb{Z}_+ \times \mathbb{Z}_+$, and p_B is some small positive number. Given p_G and p_B we ask if $\delta > 0$ may be taken small enough so that there is a positive probability of oriented percolation to infinity from the origin. We prove the answer to be positive, provided $p_G > p_c$, as stated in the next theorem, which is the main result of this article.

Theorem 1.1. *In the setup described above, let*

$$\Theta(p_G, p_B, \delta) = P(C_0 \text{ is infinite}).$$

Then, if $p_G > p_c$ and $p_B > 0$, we can find $\delta_0 = \delta_0(p_G, p_B) > 0$ so that $\Theta(p_G, p_B, \delta) > 0$ for all $\delta \leq \delta_0$. In fact, for $\delta \leq \delta_0$,

$$P(C_0 \text{ is infinite} \mid \xi) > 0 \text{ for almost all } \xi. \quad (1.1)$$

Most of our effort in the article will involve proving Theorem 1.1 for p_G close to one. The extension to any supercritical p_G is discussed at the end.

This work stems from attempts to understand and answer various questions which were naturally raised in probability, theoretical computer science and statistical physics. These questions lie on crossroads of various fields and have several quite distinct roots.

- Spatial growth processes such as percolation or contact process in random environment is a very well established topic. The situation is reasonably well understood when the environment has good space-time mixing properties. Much less is known for environments with long range dependencies. One source of inspiration is [4], where the contact process with spatial disorder persisting in time is considered. Shifting their setup to oriented percolation, the difference is that the (*good/bad*) layers in [4] are parallel to the growth direction. Our environment is somehow “orthogonal” and it generates more global effects.

It is worth comparing the situation treated here with that in [4], where survival (or percolation) is achieved by pushing the good lines to be *good enough*, given p_B and the frequency δ of bad lines. It is very simple to see that this result cannot hold in the current situation, with the layers being transversal to the growth.

- In late sixties, McCoy and Wu ([15, 16, 17, 18]) started the study of a specific class of disordered ferromagnets with random couplings that are constant along each horizontal line, for instance with randomly located layers of strongly and weakly coupled spin systems.

• A third set of questions comes from theoretical computer science. Among them, the clairvoyant scheduling problem or coordinate percolation, introduced by P. Winkler in early nineties: is it possible, in a complete graph with n vertices, to schedule two independently sampled random walks (by suitably delaying jumps), so that they never collide? This has a representation in terms of planar oriented percolation (due to Noga Alon). For results in this direction see [20, 1, 7]. The answer is negative for $n = 2$ or $n = 3$. Numerical simulations suggest a positive answer for $n \geq 4$. Recent progress in [2] gives a positive answer for n large enough.

In parallel, several questions of similar nature, such as compatibility of binary sequences, Lipschitz embedding and rough isometries of random one dimensional objects have been considered and recently answered in [3]. (See also [9, 19, 8, 12].)

The approach undertaken in [3] is, as ours, based on multi-scale analysis. While the general concept is similar, both methods are quite different in its technical execution. The scheme developed in [3] relies more on the fine probabilistic block estimates. The approach taken in our work gives a precise geometric description of the random environment, describing the global picture in terms of increasing hierarchies and inter-relation between them.

2. CONSTRUCTION OF RENORMALIZED LATTICES. STEP 1: CLUSTERS

Define $\Gamma \equiv \Gamma(\omega) = \{x \in \mathbb{Z}_+ : \xi_x = 1\}$ and label the elements of Γ in increasing order $\Gamma = \{x_j\}_{j \geq 1}$. The sequence Γ is called the *environment*.

We will build an infinite sequence $\{\mathbf{C}_k\}_{k \geq 0}$ of partitions of Γ . Each partition \mathbf{C}_k is a collection $\mathbf{C}_k = \{\mathcal{C}_{k,j}\}_{j \geq 1}$ of subsets of Γ . We call the elements of \mathbf{C}_k *clusters*. The construction depends on a parameter L , a positive integer which will be fixed later so that the property in Lemma 2.1 below is satisfied. The clusters will be constructed so as to have the properties

$$\text{each } \mathcal{C}_{k,j} \text{ is of the form } I \cap \Gamma \text{ for an interval } I, \quad (2.1)$$

and

$$\text{span}(\mathcal{C}_{k,j}) \cap \text{span}(\mathcal{C}_{k,j'}) = \emptyset \quad \text{if } j \neq j', \quad (2.2)$$

where $\text{span}(C)$ is simply the smallest interval (in \mathbb{Z}_+) that contains C . To each cluster $\mathcal{C}_{k,j}$ we will attribute a *mass*, $m(\mathcal{C}_{k,j})$, in such a way that

$$d(\mathcal{C}_{k,j}, \mathcal{C}_{k,j'}) \geq L^r, \quad (2.3)$$

if $\min\{m(\mathcal{C}_{k,j}), m(\mathcal{C}_{k,j'})\} \geq r$, for $r = 1, \dots, k$ and $j \neq j'$. Here the distance $d(D_1, D_2)$ is the usual Euclidean distance between two sets D_1 and D_2 . To each cluster $\mathcal{C}_{k,j}$ we shall further associate a number $\ell(\mathcal{C}_{k,j}) \in \{0, 1, \dots, k\}$ which will be called the *level of the cluster*. This level will satisfy

$$0 \leq \ell(\mathcal{C}_{k,j}) < m(\mathcal{C}_{k,j}). \quad (2.4)$$

Finally we construct the (limiting) partition $\mathbf{C}_\infty = \{\mathcal{C}_{\infty,j}\}_{j \geq 1}$ of Γ with the property that

$$\text{span}(\mathcal{C}_{\infty,j}) \cap \text{span}(\mathcal{C}_{\infty,j'}) = \emptyset \quad \text{if } j \neq j'. \quad (2.5)$$

To each cluster $\mathcal{C}_{\infty,j}$ of \mathbf{C}_{∞} we will attribute a mass $m(\mathcal{C}_{\infty,j})$, and a level $\ell(\mathcal{C}_{\infty,j})$, in such a way that

$$0 \leq \ell(\mathcal{C}_{\infty,j}) < m(\mathcal{C}_{\infty,j}), \quad (2.6)$$

and the following property holds:

$$d(\mathcal{C}_{\infty,j}, \mathcal{C}_{\infty,j'}) \geq L^r, \text{ if } \min\{m(\mathcal{C}_{\infty,j}), m(\mathcal{C}_{\infty,j'})\} \geq r, \text{ for } r \geq 1 \text{ and } j \neq j', \quad (2.7)$$

or equivalently,

$$d(\mathcal{C}_{\infty,j}, \mathcal{C}_{\infty,j'}) \geq L^{\min\{m(\mathcal{C}_{\infty,j}), m(\mathcal{C}_{\infty,j'})\}}, \text{ for } j \neq j'. \quad (2.8)$$

The construction.

Recall that $\Gamma \equiv \Gamma(\omega) = \{x \in \mathbb{Z}_+ : \xi_x = 1\}$, and that the elements of Γ are labeled in increasing order. $\Gamma = \{x_j\}_{j \geq 1}$ with $x_1 < x_2 < \dots$.

Level 0. The clusters of level 0 are just the subsets of Γ of cardinality one. We take $\mathcal{C}_{0,j} = \{x_j\}$ and attribute a unit mass to each such cluster. That is, $m(\mathcal{C}_{0,j}) = 1$ and $\ell(\mathcal{C}_{0,j}) = 0$. Set $\mathbf{C}_{0,0} = \mathbf{C}_0 = \{\mathcal{C}_{0,j}\}_{j \geq 1}$. Further define $\alpha(\mathcal{C}) = \omega(\mathcal{C}) = x$ when $\mathcal{C} = \{x\}$ is a cluster of level 0.

Level 1. We say that $x_i, x_{i+1}, \dots, x_{i+n-1}$ form a *maximal 1-run of length $n \geq 2$* if

$$x_{j+1} - x_j < L, \quad j = i, \dots, i+n-2,$$

and

$$x_{j+1} - x_j \geq L \quad \begin{cases} \text{for } j = i-1, j = i+n-1, & \text{if } i > 1 \\ \text{for } j = i+n-1, & \text{if } i = 1. \end{cases}$$

The level 0 clusters $\{x_i\}, \{x_{i+1}\}, \dots, \{x_{i+n-1}\}$ will be called *constituents* of the run. Note that there are no points in Γ between two consecutive points of a maximal 1-run. Also note that if $x_{j+1} - x_j \geq L$ and $x_j - x_{j-1} \geq L$, then x_j does not appear in any maximal 1-run of length at least 2.

For any pair of distinct maximal runs, r' and r'' say, all clusters in r' lie to the left of all clusters in r'' or vice versa. It therefore makes sense to label the consecutive maximal 1-runs of length at least 2 in increasing order of appearance: r_1^1, r_2^1, \dots . It is immediate that P -a.s. all runs are finite, and that infinitely many such runs exist. We write $r_i^1 = r_i^1(x_{s_i}, x_{s_i+1}, \dots, x_{s_i+n_i-1})$, $i = 1, \dots$, if the i -th run consists of $x_{s_i}, x_{s_i+1}, \dots, x_{s_i+n_i-1}$. Note that $n_i \geq 2$ and $s_i + n_i \leq s_{i+1}$ for each i . The set

$$\mathcal{C}_i^1 = \{x_{s_i}, x_{s_i+1}, \dots, x_{s_i+n_i-1}\}$$

is called a *level 1-cluster*, i.e., $\ell(\mathcal{C}_i^1) = 1$. We attribute to \mathcal{C}_i^1 the mass given by its cardinality:

$$m(\mathcal{C}_i^1) = n_i.$$

The points

$$\alpha_i^1 = x_{s_i} \quad \text{and} \quad \omega_i^1 = x_{s_i+n_i-1}$$

are called, respectively, the *start-point* and *end-point* of the run, as well as of the cluster \mathcal{C}_i^1 . To avoid confusion we sometimes write more explicitly $\alpha(\mathcal{C}_i^1)$, $\omega(\mathcal{C}_i^1)$.

By $\mathbf{C}_{1,1}$ we denote the set of clusters of level 1. Let $\mathbf{C}'_{0,1} = \{\{x_i\} : x_i \in \Gamma \setminus \mathcal{C} \text{ for all } \mathcal{C} \in \mathbf{C}_{1,1}\}$ and $\mathbf{C}_1 = \mathbf{C}_{1,1} \cup \mathbf{C}'_{0,1}$. Note that $\mathbf{C}_{1,1}$ and $\mathbf{C}'_{0,1}$ consist of level 1 and level 0 clusters, respectively, and that the union of all points in these clusters is exactly Γ . We label the elements of \mathbf{C}_1 in increasing order as $\mathcal{C}_{1,j}, j \geq 1$. For later use we also define $\mathbf{C}_{0,1} = \mathbf{C}_0$. **Notation.** In our notation \mathcal{C}_j^1 denotes the j^{th} level 1-cluster, and $\mathcal{C}_{1,j}$ denotes the j^{th} element in \mathbf{C}_1 (always in increasing order).

Level $k+1$. Let $k \geq 1$ and assume that the partitions $\mathbf{C}_{k'} = \{\mathcal{C}_{k',j} : j \geq 1\}$, and the masses of the $\mathcal{C}_{k',j}$ have already been defined for $k' \leq k$, and satisfy the properties (2.1)-(2.3) and that \mathbf{C}_k consists of clusters \mathcal{C} of levels $\ell \in \{0, 1, \dots, k\}$, i.e.,

$$\mathbf{C}_k \subset \bigcup_{\ell=0}^k \mathbf{C}_{\ell,\ell}, \quad (2.9)$$

where for $\ell \geq 0$, $\mathbf{C}_{\ell,\ell}$ is the set of level ℓ clusters. We assume, as before, that the labeling goes in increasing order of appearance. Define

$$\mathbf{C}_{k,k+1} = \{\mathcal{C} \in \mathbf{C}_k : m(\mathcal{C}) \geq k+1\}. \quad (2.10)$$

Notice that $\mathbf{C}_{1,2} = \mathbf{C}_{1,1}$ and $\mathbf{C}_{k,k+1} \subseteq \bigcup_{\ell=1}^k \mathbf{C}_{\ell,\ell}$, if $k \geq 1$.

In the previous enumeration of \mathbf{C}_k , let $j_1 < j_2 < \dots$ be the labels of the clusters in $\mathbf{C}_{k,k+1}$, so that $\mathbf{C}_{k,k+1} = \{\mathcal{C}_{k,j_1}, \mathcal{C}_{k,j_2}, \dots\}$. In $\mathbf{C}_{k,k+1}$ we consider consecutive maximal $(k+1)$ -runs, where we say that the clusters $\mathcal{C}_{k,j_s}, \mathcal{C}_{k,j_{s+1}}, \dots, \mathcal{C}_{k,j_{s+n-1}} \in \mathbf{C}_{k,k+1}$ form a *maximal* $(k+1)$ -run of length $n \geq 2$ if:

$$d(\mathcal{C}_{k,j_i}, \mathcal{C}_{k,j_{i+1}}) < L^{k+1}, \quad i = s, \dots, s+n-2,$$

and in addition

$$d(\mathcal{C}_{k,j_i}, \mathcal{C}_{k,j_{i+1}}) \geq L^{k+1} \quad \begin{cases} \text{for } i = s-1, i = s+n-1, & \text{if } j_s > 1 \\ \text{for } i = s+n-1, & \text{if } j_s = 1. \end{cases}$$

Again it is immediate that P -a.s. all $(k+1)$ -runs are finite and that infinitely many such runs exist. Again we can label them in increasing order and write $r_i^{k+1} = r_i^{k+1}(\mathcal{C}_{k,j_{s_i}}, \mathcal{C}_{k,j_{s_i+1}}, \dots, \mathcal{C}_{k,j_{s_i+n_i-1}})$ for the i -th $(k+1)$ -run, for suitable s_i, n_i such that $n_i \geq 2$ and $s_i + n_i \leq s_{i+1}$ for all i . (s_i, n_i have nothing to do with those in the previous steps of the construction.) We set

$$\alpha_i^{k+1} = \alpha(\mathcal{C}_{k,j_{s_i}}) \quad \text{and} \quad \omega_i^{k+1} = \omega(\mathcal{C}_{k,j_{s_i+n_i-1}}),$$

and call these the start-point and end-point of the run, respectively. We define the span of the run

$$\text{span}(r_i^{k+1}) = [\alpha_i^{k+1}, \omega_i^{k+1}],$$

and associate to it a cluster \mathcal{C}_i^{k+1} of level $k+1$, defined as

$$\mathcal{C}_i^{k+1} = \text{span}(r_i^{k+1}) \cap \Gamma.$$

In this case, the clusters $\mathcal{C}_{k,j_{s_i}}, \mathcal{C}_{k,j_{s_i+1}}, \dots, \mathcal{C}_{k,j_{s_i+n_i-1}}$ are called *constituents* of \mathcal{C}_i^{k+1} . To the cluster \mathcal{C}_i^{k+1} we attribute the mass $m(\mathcal{C}_i^{k+1})$ by the following rule:

$$m(\mathcal{C}_i^{k+1}) = m(\mathcal{C}_{k,j_{s_i}}) + \sum_{s=s_i+1}^{s_i+n_i-1} (m(\mathcal{C}_{k,j_s}) - k) = \sum_{s=s_i}^{s_i+n_i-1} m(\mathcal{C}_{k,j_s}) - k(n_i - 1). \quad (2.11)$$

The points α_i^{k+1} and ω_i^{k+1} will also be called, respectively, *start-* and *end-point* of the cluster \mathcal{C}_i^{k+1} , and are also written as $\alpha(\mathcal{C}_i^{k+1})$ and $\omega(\mathcal{C}_i^{k+1})$.

By $\mathbf{C}_{k+1,k+1}$ we denote the set of all level $(k+1)$ clusters. Take $\mathbf{C}'_{k,k+1} = \{\mathcal{C} \in \mathbf{C}_k : \mathcal{C} \cap \text{span}(r_i^{k+1}) = \emptyset, i = 1, 2, \dots\}$. Finally we define $\mathbf{C}_{k+1} := \mathbf{C}_{k+1,k+1} \cup \mathbf{C}'_{k,k+1}$. We label the elements of \mathbf{C}_{k+1} as $\mathcal{C}_{k+1,j}, j \geq 1$, in increasing order. Note that a cluster in \mathbf{C}_k is also a cluster in \mathbf{C}_{k+1} if and only if it is disjoint from the span of each maximal $(k+1)$ -run of length at least 2. Thus \mathbf{C}_{k+1} may contain some clusters of level no more than k , but some clusters (of level $\leq k$) in \mathbf{C}_k no longer appear in \mathbf{C}_{k+1} (or any \mathbf{C}_{k+j} with $j \geq 1$).

Note also that in the formation of a cluster of level $(k+1)$, clusters of mass at most k might be incorporated while taking the span of a $(k+1)$ -run; they form what we call *dust* or *porous medium* of level at most $k-1$ in between the constituents, which have mass at least $k+1$.

This describes the construction of the \mathbf{C}_k . We next show by induction that

$$\mathbf{C}_k \text{ is a partition of } \Gamma \text{ and } \mathbf{C}_k \text{ is a refinement of } \mathbf{C}_{k+1} \quad (2.12)$$

for $k \geq 0$. This is clear for $k = 0$, since \mathbf{C}_0 is just the partition of Γ into singletons. If we already know (2.12) for $0 \leq k \leq K$, then it follows also for $k = K+1$ from the fact that clusters in \mathbf{C}_{k+1} are formed from the clusters in \mathbf{C}_k by combining the consecutive clusters between the start- and end-point of a maximal $(k+1)$ -run into one cluster. Thus it takes a number of successive clusters in \mathbf{C}_k and combines them into one cluster. This establishes (2.12) for all k .

The definition of \mathbf{C}_k shows that

$$\mathbf{C}_{k,k} \subset \mathbf{C}_k \subset \mathbf{C}_{k,k} \cup \mathbf{C}'_{k-1,k} \subset \mathbf{C}_{k,k} \cup \mathbf{C}_{k-1},$$

from which we obtain by induction that (2.9) holds, as well as

$$\ell(\mathcal{C}) = \min\{k : \mathcal{C} \in \mathbf{C}_k\}$$

for any $\mathcal{C} \in \cup_{k \geq 1} \mathbf{C}_k$.

We use induction once more to show that for any $k \geq 0$

$$m(\mathcal{C}) \geq \ell(\mathcal{C}) + 1 \text{ for any } \mathcal{C} \in \cup_{0 \leq \ell \leq k} \mathbf{C}_\ell, \quad (2.13)$$

and if \mathcal{C}_i^{k+1} is formed from the constituents $\mathcal{C}_{k,j_{s_i}}, \dots, \mathcal{C}_{k,j_{s_i+n_i-1}}$ with $n_i \geq 2$, then

$$m(\mathcal{C}_i^{k+1}) \geq \max_{s_i \leq s \leq s_i+n_i-1} m(\mathcal{C}_{k,j_s}) + n_i - 1 > \max_{s_i \leq s \leq s_i+n_i-1} m(\mathcal{C}_{k,j_s}). \quad (2.14)$$

Indeed, (2.13) trivially holds for $k = 0$. Moreover, if (2.13) holds for $k \leq K$, then (2.14) for $k = K$ follows from the rule (2.11) (and $n_i \geq 2$). In turn, (2.14) and (2.13) for $k \leq K$ imply

$$m(\mathcal{C}_i^{k+1}) \geq \max_{s_i \leq s \leq s_i + n_i - 1} m(\mathcal{C}_{k,j_s}) + 1, \quad (2.15)$$

and hence also (2.13) for $k = K + 1$.

So far we have shown that \mathbf{C}_{k+1} is a partition of Γ which satisfies (2.1) and (2.2) with k replaced by $k + 1$ (by the definition of $\mathbf{C}'_{k,k+1}$ and induction on k). We next show by an indirect proof that this is also true for (2.3). It is convenient to first prove the following claim:

Claim. If $t \geq 1$, $\mathcal{C} \in \cup_{j \geq 0} \mathbf{C}_{t+j}$ and $\ell(\mathcal{C}) \leq t$, then we have (see definition (2.10))

$$\mathcal{C} \in \mathbf{C}_{s,s+1} \text{ for } \ell(\mathcal{C}) \leq s \leq (m(\mathcal{C}) - 1) \wedge t. \quad (2.16)$$

To see this, define \hat{s} as the smallest $s \geq \ell(\mathcal{C})$ for which $\mathcal{C} \notin \mathbf{C}_{s,s+1}$, and assume that $\hat{s} \leq (m(\mathcal{C}) - 1) \wedge t$. Then $m(\mathcal{C}) \geq \hat{s} + 1$, so that we must have $\mathcal{C} \notin \mathbf{C}_{\hat{s}}$. But also $\mathcal{C} \in \mathbf{C}_{\hat{s}-1,\hat{s}} \subseteq \mathbf{C}_{\hat{s}-1}$. (Note that $\hat{s} = \ell(\mathcal{C})$ cannot occur, because one always has $\mathcal{C} \in \mathbf{C}_{\ell,\ell+1}$ for $\ell = \ell(\mathcal{C})$, by virtue of (2.13).) Then it must be the case that \mathcal{C} intersects $\text{span}(r_i^{\hat{s}})$ for some i . In fact, by our construction, \mathcal{C} must then be a constituent of some cluster in $\mathbf{C}_{\hat{s}}$ corresponding to a maximal \hat{s} -run of length at least 2. But then \mathcal{C} does not appear in $\mathbf{C}_{\hat{s}+j}$ for any $j \geq 0$, and in particular $\mathcal{C} \notin \cup_{j \geq 0} \mathbf{C}_{t+j}$, contrary to our assumption. Thus, $\hat{s} \leq (m(\mathcal{C}) - 1) \wedge t$ is impossible and our claim must hold.

We now turn to the proof of (2.3). This is obvious for $k = 0$ or $k = 1$. Assume then that (2.3) has been proven for some $k \geq 1$. Assume further, to derive a contradiction, that \mathcal{C}' and \mathcal{C}'' are two distinct clusters in \mathbf{C}_{k+1} such that $\min\{m(\mathcal{C}'), m(\mathcal{C}'')\} \geq r$ but $d(\mathcal{C}', \mathcal{C}'') < L^r$ for some $r \leq k + 1$. Without loss of generality we take $r = m(\mathcal{C}') \wedge m(\mathcal{C}'') \wedge (k + 1)$. Let \mathcal{C}' and \mathcal{C}'' have level ℓ' and ℓ'' , respectively. Since these clusters belong to \mathbf{C}_{k+1} we must have $\max(\ell', \ell'') \leq k + 1$. For the sake of argument, let $\ell' \leq \ell''$. If $\ell' = \ell'' = k + 1$, then $d(\mathcal{C}', \mathcal{C}'') \geq L^{k+1}$, because, by construction, two distinct clusters of level $k + 1$ have distance at least L^{k+1} . In this case we don't have $d(\mathcal{C}', \mathcal{C}'') < L^r$, so that we may assume $\ell' < k + 1$.

Now first assume that $r - 1 \geq \max(\ell', \ell'') = \ell''$. Since $r - 1 \leq k$ we then have by (2.16) (with $t = k$) that \mathcal{C}' and \mathcal{C}'' both belong to $\mathbf{C}_{r-1,r}$. If the distance from \mathcal{C}' to the nearest cluster in $\mathbf{C}_{r-1,r}$ is less than L^r , then \mathcal{C}' will be a constituent of a cluster of level r and \mathcal{C}' will not be an element of \mathbf{C}_{k+1} . Thus it must be the case that the distance from \mathcal{C}' to the nearest cluster in $\mathbf{C}_{r-1,r}$ is at least L^r . A fortiori, $d(\mathcal{C}', \mathcal{C}'') \geq L^r$. This contradicts our choice of $\mathcal{C}', \mathcal{C}''$.

The only case left to consider is when $r - 1 < \max(\ell', \ell'') = \ell''$. Since $r - 1 = (m(\mathcal{C}') - 1) \wedge (m(\mathcal{C}'') - 1) \wedge k \geq \ell' \wedge \ell'' \wedge k = \ell'$ (by (2.13); recall that $\ell' < k + 1$ now) this means $\ell' \leq r - 1 < \ell''$. We still have as in the last paragraph that $\mathcal{C}' \in \mathbf{C}_{r-1,r}$, and that the distance between \mathcal{C}' and the nearest cluster in $\mathbf{C}_{r-1,r}$ is at least L^r . By (2.2) $\text{span}(\mathcal{C}')$ and $\text{span}(\mathcal{C}'')$ have to be disjoint. For the sake of argument let us further assume that \mathcal{C}' lies to the left of \mathcal{C}'' , that is, $\omega(\mathcal{C}') < \alpha(\mathcal{C}'')$. We claim that $\alpha(\mathcal{C}'') = \alpha(\mathcal{C})$ for some cluster $\mathcal{C} \in \mathbf{C}_{r-1,r}$. Indeed, the start-point of a cluster of level $\tilde{\ell} \geq 2$ equals the start-point of one of its constituents,

which belongs to $\mathbf{C}_{\tilde{\ell}-1, \tilde{\ell}} \subseteq \mathbf{C}_{\tilde{\ell}-1}$. Repetition of this argument shows that $\alpha(\mathcal{C}'')$ is also the start-point of a cluster \mathcal{C} which is a constituent of some cluster $\widehat{\mathcal{C}}$ such that $s := \ell(\mathcal{C}) \leq r-1$ but $t+1 := \ell(\widehat{\mathcal{C}}) \geq r$. In particular, $\mathcal{C} \in \mathbf{C}_{t,t+1}$, so that $\mathcal{C} \in \mathbf{C}_t$ and $m(\mathcal{C}) \geq t+1 \geq r$. Thus $\ell(\mathcal{C}) \leq r-1 \leq (m(\mathcal{C})-1) \wedge t$. It then follows from (2.16) that $\mathcal{C} \in \mathbf{C}_{r-1,r}$. As in the preceding case we then have

$$\begin{aligned} d(\mathcal{C}', \mathcal{C}'') &\geq \alpha(\mathcal{C}'') - \omega(\mathcal{C}') = \alpha(\mathcal{C}) - \omega(\mathcal{C}') \\ &\geq \text{the distance from } \mathcal{C}' \text{ to the nearest cluster in } \mathbf{C}_{r-1,r} \geq L^r. \end{aligned}$$

Of course the inequality for $d(\mathcal{C}', \mathcal{C}'')$ remains valid if \mathcal{C}' lies to the right of \mathcal{C}'' , so that we have arrived at a contradiction in all cases, and (2.3) with k replaced by $k+1$ must hold. This completes the proof of (2.3).

Construction of \mathbf{C}_∞ . Observe that each $x \in \Gamma$ may belong to clusters of several levels, but not to different clusters of the same level (see (2.2)). If \mathcal{C}' and \mathcal{C}'' are two clusters of levels ℓ' and ℓ'' , respectively, with $\ell' < \ell''$, then

$$\text{span}(\mathcal{C}') \cap \text{span}(\mathcal{C}'') \neq \emptyset \text{ implies } \text{span}(\mathcal{C}') \subseteq \text{span}(\mathcal{C}''). \quad (2.17)$$

There will even exist a sequence $\mathcal{C}_0 = \mathcal{C}', \mathcal{C}_1, \dots, \mathcal{C}_s, \mathcal{C}_{s+1} = \mathcal{C}''$ such that \mathcal{C}_i is a constituent of \mathcal{C}_{i+1} , $0 \leq i \leq s$. This follows from the fact that each \mathbf{C}_k is a partition of Γ and that \mathbf{C}_k is a refinement of \mathbf{C}_{k+1} . In fact, each element of \mathbf{C}_{k+1} is obtained by combining several consecutive elements of \mathbf{C}_k . (We allow here that an element of \mathbf{C}_k is already an element of \mathbf{C}_{k+1} by itself.) In turn, we see then from (2.15) that $m(\mathcal{C}'') > m(\mathcal{C}')$. In particular, no point belongs to two different clusters with the same mass. We shall use this fact in the proof of the next lemma.

We define the random index

$$\kappa(x) = \sup\{\ell: x \in \mathcal{C} \text{ for some } \mathcal{C} \in \mathbf{C}_{\ell,\ell}\}.$$

If we allow the value ∞ for $\kappa(x)$, then this index is always well defined, since each $x \in \Gamma$ belongs at least to the cluster $\{x\}$ of level 0.

Lemma 2.1. *For $\delta > 0$ and $3 \leq L < (64\delta)^{-1/2}$ we have a.s. $\kappa(x) < \infty$ for all $x \in \Gamma$.*

Before proving this lemma we show how to use it for the construction of \mathbf{C}_∞ . Lemma 2.1 tells that for each $x \in \Gamma$, there exists a cluster of level $\kappa(x) \in \mathbb{Z}_+$ which contains x . This cluster is unique, since the elements of $\mathbf{C}_{k,k}$ are pairwise disjoint. We call it the *maximal cluster of x* and denote it by \mathcal{D}_x . Moreover, for $x, x' \in \Gamma$, if $x' \in \mathcal{D}_x$, then $\kappa(x) = \kappa(x')$ and $\mathcal{D}_x = \mathcal{D}_{x'}$. Indeed, $\kappa(x) \neq \kappa(x')$ would contradict (2.17) and the definition of κ , while $\kappa(x) = \kappa(x')$ but $\mathcal{D}_x \neq \mathcal{D}_{x'}$ is impossible by (2.2).

Take $\hat{x}_1 = x_1 \in \Gamma = \{x_j\}_{j \geq 1}$ and define $\mathcal{C}_{\infty,1} = \mathcal{D}_{\hat{x}_1}$. Having defined $\mathcal{C}_{\infty,j} = \mathcal{D}_{\hat{x}_j}$ for $j = 1, \dots, k$, we set $\hat{x}_{k+1} = \min\{x_j \in \Gamma: x_j \notin \cup_{i=1}^k \mathcal{C}_{\infty,i}\}$, and $\mathcal{C}_{\infty,k+1} = \mathcal{D}_{\hat{x}_{k+1}}$. Define $\mathbf{C}_\infty = \{\mathcal{C}_{\infty,k}\}_{k \geq 1}$. Clearly, $\Gamma = \cup_{k \geq 1} \mathcal{C}_{\infty,k}$. It is also routine to check that \mathbf{C}_∞ satisfies (2.5) and (2.6). As for (2.7), this follows from (2.3) and the fact that $\mathcal{D}_x \in \mathbf{C}'_{k,k+1} \subseteq \mathbf{C}_{k+1}$ for all $k \geq \kappa(x)$ (by the definitions of $\kappa(x)$ and $\mathbf{C}'_{k,k+1}$).

Proof of Lemma 2.1. $\kappa(x) = +\infty$ can occur only if there exists an infinite increasing subsequence of indices $\{k_i\}_{i \geq 1}$ such that the point x becomes “incorporated” into some cluster of level k_i for all $i \geq 1$. We will show that

$$P(x \text{ belongs to an infinite sequence of clusters}) = 0. \quad (2.18)$$

Notice that each cluster of level k necessarily has mass at least $k + 1$ and no point belongs to two different clusters of the same mass, as observed above. Setting

$$A_k(x) = [x \text{ belongs to a cluster of mass } k], \quad (2.19)$$

we shall show that for each fixed x

$$P(A_k(x) \text{ i.o. in } k) = 0, \quad (2.20)$$

which will prove (2.18).

We will carry out the proof in two steps. All constants c_i below are strictly positive and independent of k . First we estimate the probability that a given point $z \in \mathbb{Z}_+$ is the start-point of a cluster of mass $k \geq 2$. Specifically, we show that

$$P(\exists \mathcal{C} \in \bigcup_{\ell \geq 1} \mathbf{C}_\ell : \alpha(\mathcal{C}) = z, m(\mathcal{C}) = k) \leq c_1 e^{-c_2 k} \quad (2.21)$$

for some strictly positive constants $c_1, c_2 > 0$ and for each k and each z . In fact we can take $c_2 > \log L$ so that

$$c_1(L^k + 1)e^{-c_2 k} \leq 2c_1 e^{-c_3 k} \quad (2.22)$$

for some constant $c_3 > 0$. This is the most involved part of the proof. In the second step of the proof we show that if $\mathcal{C} \in \bigcup_{\ell \geq 1} \mathbf{C}_\ell$ and $m(\mathcal{C}) = k$, then

$$\text{diam}(\mathcal{C}) \leq 3L^{k-1}. \quad (2.23)$$

Due to (2.23) we will have the following inclusion:

$$A_k(x) \subseteq [\exists z \in [x - L^k, x] : \alpha(\mathcal{C}) = z, \text{ for some } \mathcal{C} \in \bigcup_{\ell \geq 1} \mathbf{C}_\ell \text{ and } m(\mathcal{C}) = k]. \quad (2.24)$$

From (2.21), (2.22) and (2.24) we will have

$$P(A_k(x)) \leq c_1(L^k + 1)e^{-c_2 k} \leq 2c_1 e^{-c_3 k}, \quad (2.25)$$

which, by the Borel-Cantelli lemma, gives (2.20), and so (2.18).

Let us now prove (2.21), where $k \geq 2$ and $z \in \mathbb{Z}_+$. To any given cluster $\mathcal{C} \in \bigcup_{\ell \geq 1} \mathbf{C}_\ell$ we associate a “genealogical weighted tree”. It describes the successive merging processes which lead to the creation of \mathcal{C} , i.e., it tells the levels at which some clusters form runs, merging into larger clusters and how many constituents entered each run, down to level 1, and finally the masses of such level 1 clusters. So we represent it as a tree with the root corresponding to \mathcal{C} ; the leaves correspond to clusters of level 1, which are the basic constituents at level 1. This weighted tree gives the basic information on the cluster, neglecting what was incorporated as “dust”, on the way.

More formally, we construct the tree iteratively. The root of the tree corresponds to the cluster \mathcal{C} . If this cluster is of level 1, the procedure is stopped. For notational consistency

such a tree will be called a 1-leaf tree. To the root we attribute the index 1, as well as another index which equals the mass of the cluster.

If the resulting cluster \mathcal{C} is of level $\ell > 1$, we attribute to the root the index ℓ and add to the graph n_1 edges (children) going out from the root, where $n_1 \geq 2$ is the number of constituents which form the ℓ -run leading to \mathcal{C} . Each endvertex of a newly added edge will correspond to a constituent of the run, i.e., if \mathcal{C} has constituents $\mathcal{C}_{\ell-1,i_1}, \dots, \mathcal{C}_{\ell-1,i_{n_1}} \in \mathbf{C}_{\ell-1,\ell}$, for suitable i_1, \dots, i_{n_1} , then there is a vertex at the end of an edge going out from the root corresponding to $\mathcal{C}_{\ell-1,i_j}$ for each $j = 1, \dots, n_1$. If the constituent corresponding to a given endvertex is a level 1-cluster, the procedure at this endvertex is stopped (producing a leaf on the tree), and to this leaf we attribute an index, which equals the mass of the corresponding constituent.

If a given endvertex corresponds to a cluster $\tilde{\mathcal{C}}$ of level ℓ' with $1 < \ell' < \ell$, then to this endvertex we attribute the index ℓ' , and add to the graph n_2 new edges going out of this endvertex, where n_2 is the number of constituents of $\tilde{\mathcal{C}}$ in $\mathbf{C}_{\ell'-1,\ell'}$ which make up $\tilde{\mathcal{C}}$.

The procedure continues until we reach the state that all constituents corresponding to newly added edges are level 1 clusters. In this way we obtain a tree with the following properties:

- i) each vertex of the tree has either 0 or at least two offspring; in case of 0 offspring we say that the vertex is a *leaf* of the tree. Otherwise we call it a *branch node*.
- ii) to each branch node x we attribute an index ℓ_x ; these indices are strictly decreasing to 1 along any self-avoiding path from the root to a leaf of the tree.
- iii) to each leaf is associated a mass $m \geq 1$. This defines a map

$$\gamma: \mathcal{C} \in \cup_{\ell \geq 1} \mathbf{C}_\ell \mapsto \gamma(\mathcal{C}) \equiv (\Upsilon(\mathcal{C}), \bar{l}(\mathcal{C}), \bar{m}(\mathcal{C})),$$

where $\Upsilon(\mathcal{C})$ is a finite tree with $\mathcal{L}(\Upsilon(\mathcal{C}))$ leaves and $\mathcal{N}(\Upsilon(\mathcal{C}))$ branching nodes. We use the following notation:

$\bar{l}(\mathcal{C}) = \{\ell_1(\mathcal{C}), \dots, \ell_{\mathcal{N}(\Upsilon(\mathcal{C}))}(\mathcal{C})\}$ is a multi-index with one component for each branching node of $\Upsilon(\mathcal{C})$, which indicates the level at which branches “merge” into the cluster corresponding to the node;

$\bar{m}(\mathcal{C}) = \{m_1(\mathcal{C}), \dots, m_{\mathcal{L}(\Upsilon(\mathcal{C}))}(\mathcal{C})\}$ a multi-index with one component for each leaf of $\Upsilon(\mathcal{C})$, which gives to the mass of the cluster corresponding to the leaf;

$\bar{n}(\mathcal{C}) = \{n_1(\mathcal{C}), \dots, n_{\mathcal{N}(\Upsilon(\mathcal{C}))}(\mathcal{C})\}$ is a multi-index with one component for each vertex of $\Upsilon(\mathcal{C})$, which gives the degree of the vertex minus 1. Note that $\bar{n}(\mathcal{C})$ is determined by $\Upsilon(\mathcal{C})$.

To lighten the notation, we will omit the argument \mathcal{C} in situations where confusion is unlikely. Thus we occasionally write $\gamma(\mathcal{C}) \equiv (\Upsilon, \bar{l}, \bar{m})$ instead of $(\Upsilon(\mathcal{C}), \bar{l}(\mathcal{C}), \bar{m}(\mathcal{C}))$.

In order to prove (2.21) we decompose the event

$$[\exists \mathcal{C} \in \cup_{\ell \geq 1} \mathbf{C}_\ell: \alpha(\mathcal{C}) = z, m(\mathcal{C}) = k] \quad (2.26)$$

according to the possible values for $\gamma(\mathcal{C})$; we shall abbreviate the number of leaves of $\Upsilon(\mathcal{C})$ by \mathcal{L} . Since the resulting cluster \mathcal{C} , obtained after all merging process “along the tree”, has

mass k , it imposes the following relation between the multi-indices \bar{m} and \bar{l} :

$$\sum_{i=1}^{\mathcal{L}} m_i - \sum_{j=1}^{\mathcal{N}} (n_j - 1)(\ell_j - 1) = k \quad (2.27)$$

Here the first sum runs over all leaves, while the second sum runs over all branching nodes. This relation follows from (2.11) by induction on the number of vertices, by writing the tree as the “union” of the root and the subtrees which remain after removing the root. We note that Υ also has to satisfy

$$\sum_{j=1}^{\mathcal{N}} (n_j - 1) = \mathcal{L} - 1, \quad (2.28)$$

because it is a tree, as one easily sees by induction on the number of leaves. This implies the further restriction

$$\sum_{i=1}^{\mathcal{L}} m_i \geq k + \mathcal{L} - 1,$$

because $\ell_j \geq 2$ in each term of the second sum in (2.27) (recall that we stop our tree construction at each node corresponding to a cluster of level 1). Thus the probability of the event in (2.26) equals to

$$\sum_{r \geq 1} \sum_{\substack{\Upsilon: \\ \mathcal{L}(\Upsilon)=r}} \sum_{\bar{l}, \bar{m}}^{\Upsilon} P(\exists \mathcal{C} \in \cup_{\ell \geq 1} \mathbf{C}_\ell: z = \alpha(\mathcal{C}), m(\mathcal{C}) = k, \gamma(\mathcal{C}) = (\Upsilon, \bar{l}, \bar{m})), \quad (2.29)$$

where the third sum $\sum_{\bar{l}, \bar{m}}^{\Upsilon}$ is taken over all possible values of \bar{l}, \bar{m} , satisfying (2.27).

A decomposition according to the value of the sum $\sum_i m_i$, shows that the expression (2.29) equals

$$\sum_{r \geq 1} \sum_{\substack{\Upsilon: \\ \mathcal{L}(\Upsilon)=r}} \sum_{s \geq r-1} \sum_{\substack{\bar{m}: \\ \sum_i m_i = k+s}} \sum_{\bar{l}} P(\exists \mathcal{C} \in \cup_{\ell \geq 1} \mathbf{C}_\ell: \alpha(\mathcal{C}) = z, m(\mathcal{C}) = k, \gamma(\mathcal{C}) = (\Upsilon, \bar{l}, \bar{m})), \quad (2.30)$$

the sum $\sum_{\bar{l}}$ being taken over possible choices of \bar{l} such that $\sum_j (n_j - 1)(\ell_j - 1) = s$. The multiple sum in (2.30) can be bounded from above by

$$\sum_{r \geq 1} \sum_{\substack{\Upsilon: \\ \mathcal{L}(\Upsilon)=r}} \sum_{s \geq r-1} \sum_{\substack{\bar{m}: \\ \sum_i m_i = k+s}} \sum_{\bar{l}} \delta^{k+s} L^{k+2s}. \quad (2.31)$$

Indeed, for fixed z, k and $(\Upsilon, \bar{l}, \bar{m})$, the probability

$$P(\exists \mathcal{C}: \alpha(\mathcal{C}) = z, m(\mathcal{C}) = k, \gamma(\mathcal{C}) = (\Upsilon, \bar{l}, \bar{m}))$$

is easily estimated by the following argument: the probability to find a level 1 cluster of mass m_i which corresponds to some leaf of the tree, and which starts at a given point x , is bounded from above by $\delta^{m_i} L^{m_i-1}$. Indeed, such a cluster has to come from a maximal level 1 run $x_s, x_{s+1}, \dots, x_{s+m_i-1}$ of elements of Γ , with $x_s = x$ and $x_{j+1} - x_j \leq L$ for $j = s, \dots, s+m_i-2$.

The number of choices for such a run is at most L^{m_i-1} , and given the x_j , the probability that they all lie in Γ is δ^{m_i} . Similarly, the probability to find two level 1 clusters of mass m_{i_1} and m_{i_2} which merge at level ℓ_j can be bounded above by $\delta^{m_{i_1}} L^{m_{i_1}-1} \delta^{m_{i_2}} L^{m_{i_2}-1} L^{\ell_j}$. The factor L^{ℓ_j} here is an upper bound for the number of choices for the distance between the two clusters; if they are to merge at level ℓ_j , their distance can be at most L^{ℓ_j} . Iterating this argument we get that

$$P(\exists \mathcal{C} \in \cup_{\ell \geq 1} \mathbf{C}_\ell : \alpha(\mathcal{C}) = z, m(\mathcal{C}) = k, \gamma(\mathcal{C}) = (\Upsilon, \bar{l}, \bar{m})) \leq \delta^{\sum_i m_i} L^{\sum_i (m_i-1)} L^{\sum_j (n_j-1)\ell_j},$$

and taking into account that

$$\sum_i (m_i - 1) + \sum_j (n_j - 1)\ell_j = k + s + s - r + \sum_j (n_j - 1),$$

as well as (2.28), we get the bound (2.31).

The number of terms in the sums over \bar{m} and \bar{l} in (2.31) are respectively bounded by 2^{k+s} and 2^s (since $\sum_j (\ell_j - 1) \leq \sum_j (n_j - 1)(\ell_j - 1) = s$ and $\ell_j \geq 2$). Thus we can bound (2.31) from above by

$$\begin{aligned} & \sum_{r \geq 1} \sum_{\Upsilon: \mathcal{L}(\Upsilon)=r} \sum_{s \geq r-1} 2^{k+s} 2^s \delta^{k+s} L^{k+2s} \\ & \leq (2\delta L)^k \sum_{r \geq 1} \sum_{\Upsilon: \mathcal{L}(\Upsilon)=r} \sum_{s \geq r-1} (4\delta L^2)^s \\ & \leq (2\delta L)^k \sum_{r \geq 1} \sum_{\Upsilon: \mathcal{L}(\Upsilon)=r} \frac{(4\delta L^2)^{r-1}}{1-4\delta L^2}, \end{aligned} \quad (2.32)$$

provided we take $4\delta L^2 < 1$. Now the number of planted plane trees of u vertices is at most 4^u (see [10]). Our trees have r leaves, but all vertices which are not leaves have degree at least 3 (except, possibly, the root). Thus, by virtue of (2.28), these trees have at most $2r$ vertices. The number of possibilities for Υ in the last sum is therefore at most $\sum_{u=r+1}^{2r} 4^u \leq \frac{4}{3} 4^{2r} \leq 2 \cdot 4^{2r}$. It follows that (2.32) is further bounded by

$$2 \frac{(2\delta L)^k}{1-4\delta L^2} \sum_{r \geq 1} 4^{2r} (4\delta L^2)^{r-1} = \frac{32(2\delta L)^k}{1-4\delta L^2} \sum_{r \geq 1} (64\delta L^2)^{r-1}.$$

If we take $64\delta L^2 < 1$, this can be bounded by

$$\frac{32(2\delta L)^k}{(1-4\delta L^2)(1-64\delta L^2)},$$

which proves (2.21) and (2.22) with $c_2 = -\log(2\delta) - \log L > \log L$ for our choice of δ, L . It remains to show (2.23). It is trivially correct for $k = 1$; in fact a cluster of mass 1 has to be a singleton by (2.14). We will use induction on k . Assume (2.23) holds for all clusters with mass at most $k-1$, where $k \geq 2$. Let \mathcal{C} be a cluster with $m(\mathcal{C}) = k$ and of level ℓ . Thus $\mathcal{C} \in \mathbf{C}_{\ell, \ell}$, and $1 \leq \ell \leq k-1$, by virtue of (2.13). If $\ell = 1$ then $\text{diam}(\mathcal{C}) \leq (k-1)L < 3L^{k-1}$, for $k \geq 2$, provided we take $L \geq 2$. If $\ell \geq 2$, then there exist $n \geq 2$, and $\mathcal{C}_{\ell-1, i_1}, \dots, \mathcal{C}_{\ell-1, i_n} \in \mathbf{C}_{\ell-1, \ell}$ such that \mathcal{C} is made up from the constituents $\mathcal{C}_{\ell-1, i_1}, \dots, \mathcal{C}_{\ell-1, i_n}$ (where, for simplicity, we have omitted the indication of the level of the constituents). If $m_j = m(\mathcal{C}_{\ell-1, i_j})$, then $m_j \geq \ell$ (by

(2.13)), and from (2.11) we see that $m_j \leq k - n + 1$ for each j . From this and the induction hypothesis we get

$$\text{diam}(\mathcal{C}) \leq \sum_{j=1}^n \text{diam}(\mathcal{C}_{\ell-1, i_j}) + (n-1)L^\ell < 3nL^{k-n} + (n-1)L^{k-n+1} \leq 3L^{k-1} \quad (2.33)$$

for all $L \geq 3$ and $n \geq 2$. This proves (2.23) and the lemma. \square

Note that \mathbf{C}_∞ depends on the collection Γ only. We shall occasionally write $\mathbf{C}_\infty(\gamma)$ for the partition \mathbf{C}_∞ at a sample point with $\Gamma = \gamma$. We further define

$$\chi(\gamma) = \inf\{k \geq 0: d(\mathcal{C}, 0) \geq L^{m(\mathcal{C})} \text{ for all } \mathcal{C} \in \mathbf{C}_\infty(\gamma) \text{ with } m(\mathcal{C}) > k\}, \quad (2.34)$$

and set $\chi(\gamma) = \infty$ if the above set is empty.

The preceding proof has the following corollary:

Corollary 2.2. *Under the conditions of Lemma 2.1 we have $\chi(\gamma) < +\infty$ a.s.*

In fact, (2.21) and (2.22) show that

$$P(\exists \mathcal{C}_{\infty, j}: m(\mathcal{C}_{\infty, j}) > k, d(\mathcal{C}_{\infty, j}, 0) < L^{m(\mathcal{C}_{\infty, j})}) \leq \sum_{m > k} c_1 L^m e^{-c_2 m} \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (2.35)$$

In the next lemma $P(A|\gamma)$ denotes the value of the conditional probability of an event A with respect to the σ -field generated by Γ on the event $\{\Gamma = \gamma\}$.

Lemma 2.3. *Under the conditions of Lemma 2.1 it suffices for Theorem 1.1 to prove that*

$$P(C_0 \text{ is infinite}|\gamma) > 0 \text{ a.s. on the event } \{\chi(\gamma) = 0\}. \quad (2.36)$$

Proof. We shall show here that

$$P(\chi(\Gamma) = 0) > 0. \quad (2.37)$$

Together with (2.36), this will imply that

$$\begin{aligned} P(C_0 \text{ is infinite}) &\geq P(C_0 \text{ is infinite and } \chi(\Gamma) = 0) \\ &= \int_{\{\chi(\gamma)=0\}} P(C_0 \text{ is infinite}|\gamma) P(\Gamma \in d\gamma) > 0, \end{aligned}$$

and hence $P(C_0 \text{ is infinite}) > 0$. This will also prove (1.1), because $\{P(C_0 \text{ is infinite}|\xi) > 0\}$ is a tail event (in ξ), and (2.36), (2.37) imply (1.1).

Thus, we merely have to prove (2.37). Now if $\chi(\gamma)$ is finite and non-zero, then there exists a unique cluster $\mathcal{C}^* \in \mathbf{C}_\infty(\gamma)$ such that $m(\mathcal{C}^*) = \chi(\gamma)$ and $d(\mathcal{C}^*, 0) < L^{\chi(\gamma)}$. The existence of \mathcal{C}^* follows at once from the definition of χ . For the uniqueness we observe that if two such clusters, say \mathcal{C}' and \mathcal{C}'' , would exist, then they would have to satisfy $d(\mathcal{C}', \mathcal{C}'') < L^{\chi(\gamma)} = L^{\min\{m(\mathcal{C}'), m(\mathcal{C}'')\}}$, which contradicts (2.8) by virtue of the assumption $\mathcal{C}', \mathcal{C}'' \in \mathbf{C}_\infty$.

We use \mathcal{C}^* to construct a new environment $\tilde{\gamma}$ corresponding to the following sequence $\{\tilde{\xi}_i\}_{i \geq 0}$ of zeroes and ones:

$$\text{if } \chi(\gamma) = 0, \text{ then } \tilde{\xi}_i = \xi_i \text{ for all } i \geq 0;$$

$$\text{if } 0 < \chi(\gamma) < \infty, \text{ then } \tilde{\xi}_i = \begin{cases} 0 & \text{if } i \leq \omega(\mathcal{C}^*) \\ \xi_i & \text{if } i > \omega(\mathcal{C}^*). \end{cases} \quad (2.38)$$

We shall now show that

$$\chi(\tilde{\gamma}) = 0. \quad (2.39)$$

Of course we only have to check this in the case $0 < \chi(\gamma) < \infty$. We claim that in this case all clusters in $\mathbf{C}_\infty(\tilde{\gamma})$ are also clusters in $\mathbf{C}_\infty(\gamma)$ (which are located in $[\omega(\mathcal{C}^*) + 1, \infty)$) and the masses of such a cluster in the two environments γ and $\tilde{\gamma}$ are the same. To see this we simply run through the construction of the clusters in $\cup_{\ell \geq 1} \mathbf{C}_\ell$ in the environment $\tilde{\gamma}$, until there arises a difference between these this construction and the construction in the environment γ . More precisely, we apply induction with respect to the level of the clusters. Clearly any cluster of level 0 in $\tilde{\gamma}$ is simply a single point of Γ which lies in $[\omega(\mathcal{C}^*) + 1, \infty)$, and has mass 1. This is also a cluster of level 0 and mass 1 in γ . Assume now that we already know that any cluster in $\tilde{\gamma}$ of level $\leq k$ is a cluster of γ of level k and located in $[\omega(\mathcal{C}^*) + 1, \infty)$ and with the same mass in γ and $\tilde{\gamma}$.

Since $\xi_i = 0$ for $i \leq \omega(\mathcal{C}^*)$ in the environment $\tilde{\gamma}$, the span of any $(k+1)$ -run in $\tilde{\gamma}$ has to be contained in $[\omega(\mathcal{C}^*) + 1, \infty)$. Therefore the span of any cluster of level $k+1$ in environment $\tilde{\gamma}$ also has to be contained in $[\omega(\mathcal{C}^*) + 1, \infty)$. In addition, since the two environments γ and $\tilde{\gamma}$ agree in this interval, a difference in the constructions or masses of some cluster of level $k+1$ can arise only because in γ there is a $(k+1)$ -run which contains clusters of level k which lie in $[\omega(\mathcal{C}^*) + 1, \infty)$ as well as clusters which intersect $[0, \omega(\mathcal{C}^*)]$. But then these clusters of level k will be constituents of a single $(k+1)$ -cluster, $\mathcal{C} \in \mathbf{C}_\infty(\gamma)$ say. $\text{span}(\mathcal{C})$ has to contain points of both $[0, \omega(\mathcal{C}^*)]$ and of $[\omega(\mathcal{C}^*) + 1, \infty)$ in γ . Consequently, $\text{span}(\mathcal{C})$ has to contain both points $\omega(\mathcal{C}^*)$ and $\omega(\mathcal{C}^*) + 1$. Since $\omega(\mathcal{C}^*) \in \mathcal{C}^*$ we then have from (2.17) that $\text{span}(\mathcal{C}^*) \subset \text{span}(\mathcal{C})$ and $\mathcal{C}^* \neq \mathcal{C}$ (because $\omega(\mathcal{C}^*) + 1 \notin \text{span}(\mathcal{C}^*)$). But no such \mathcal{C} can exist, because $\mathcal{C}^* \in \mathbf{C}_\infty$.

This establishes our last claim. Now, by definition of χ , (2.39) is equivalent to

$$d(\mathcal{C}, 0) \geq L^{m(\mathcal{C})} \quad (2.40)$$

for all clusters \mathcal{C} in $\mathbf{C}_\infty(\tilde{\gamma})$. In view of our claim this will be implied by (2.40) for all clusters \mathcal{C} in $\mathbf{C}_\infty(\gamma)$ located in $[\omega(\mathcal{C}^*), \infty)$. Now, if \mathcal{C} is such a cluster with $m(\mathcal{C}) \leq m(\mathcal{C}^*)$, then (2.40) holds, because, by virtue of (2.8),

$$d(\mathcal{C}, 0) = \alpha(\mathcal{C}) \geq \alpha(\mathcal{C}) - \omega(\mathcal{C}^*) = d(\mathcal{C}, \mathcal{C}^*) \geq L^{m(\mathcal{C})}.$$

On the other hand, if $m(\mathcal{C}) > m(\mathcal{C}^*) = \chi(\gamma)$, then the definition of χ shows that we have $\alpha(\mathcal{C}) \geq L^{m(\mathcal{C})}$. This proves (2.40) in all cases, and therefore also proves (2.39).

We now have

$$1 = P(\chi(\Gamma) < \infty) \leq P(\chi(\Gamma) = 0) + \sum_{n=0}^{\infty} P(\omega(\mathcal{C}^*) = n, \chi(\Gamma^{(n)}) = 0),$$

where $\omega(\mathcal{C}^*)$ is as described above with γ denoting the value of Γ , and if Γ corresponds to the sequence $\{\xi_i\}$, then $\Gamma^{(n)}$ corresponds to the sequence $\xi_i^{(n)}$ given by

$$\xi_i^{(n)} = \begin{cases} 0 & \text{if } i \leq n \\ \xi_i & \text{if } i > n. \end{cases}$$

Thus, either $P(\chi(\Gamma) = 0) > 0$ or there is some non-random $n \in \mathbb{Z}_+$ for which $P(\chi(\Gamma^{(n)}) = 0) > 0$. However,

$$\begin{aligned} P(\chi(\Gamma) = 0) &\geq P(\chi(\Gamma) = 0, \xi_i = 0 \text{ for } 0 \leq i \leq n) \\ &= P(\chi(\Gamma^{(n)}) = 0, \xi_i = 0 \text{ for } 0 \leq i \leq n) \\ &= P(\xi_i = 0 \text{ for } 0 \leq i \leq n)P(\chi(\Gamma^{(n)}) = 0) \end{aligned}$$

(since $\Gamma^{(n)}$ is determined by $(\xi_i; i > n)$). This proves the validity of (2.37) and concludes the argument. \square

3. CONSTRUCTION OF RENORMALIZED LATTICES. STEP 2: LAYERS AND SITES

From now on we restrict ourselves to the set of environments γ such that $\chi(\gamma) = 0$. In this section we construct a sequence of partitions $\{\mathbf{H}^k\}_{k \geq 0}$ of $\tilde{\mathbb{Z}}_+^2$ into horizontal layers, which will be used to define renormalized sites. As in the preceding section we will do the construction in a recursive way. The elements of \mathbf{H}^k are called *k-layers*, and they will be associated to the clusters of level at most k .

The horizontal layers in correspondence with the span of each cluster of level at most k and mass greater than k will be called *bad k-layers*. The other k -layers will be called *good k-layers*, and are further subdivided into good layers of *type 1* and of *type 2*: The good k -layers of type 1 “contain”¹ a cluster of level at most k but with mass equal to k , and the remaining good k -layers will be considered of type 2. Thus good k -layers of type 2 are “disjoint”² from the clusters of level at most k with mass at least k . Thus, bad layers are those which contain the largest (in the sense of mass) clusters of a new level. These are the most difficult for a percolation path to traverse, and have to be treated differently from the others (as we shall see later on). The good k -layers of type 2 correspond to the smallest clusters (in the sense of mass).

The good k -layers have “height” of order L^k (see (3.34) below). However, the bad layers can have different heights. Between two successive bad k -layers there are at least two good k -layers of type 2, but not necessarily any good k -layer of type 1. The total number of good k -layers between successive bad k -layers is also variable. This is reflected in the many cases covered in the following definitions. Checking some of the stated properties requires somewhat tedious verification of the different cases separately, but we have found no way to avoid this. Lemma 3.1 and the remark following it summarize some further properties of the partitions $\{\mathbf{H}^k\}$.

¹Identifying a cluster with the horizontal layer whose projection on the second coordinate is the cluster.

²idem

We shall take $L \geq 12$ and divisible by 3, with δ small enough for the assumptions of Lemma 2.1 to be verified.

Step 0. To begin, we define 0-layers in the following way: $H_j^0 = H_j = \{(x, y) \in \mathbb{Z}_+^2 : y = j\}, j \geq 0$. If $j \in \Gamma$ we say that the 0-layer H_j^0 is *bad*, and otherwise it is called *good*. These names are justified by the fact that sites which belong to good 0-layers are open with large probability (namely, p_G), and sites which belong to bad 0-layers are open with small probability (namely, p_B).

Step 1. Let $\mathbf{C}_1 = \{\mathcal{C}_{1,j}\}_{j \geq 1}$. We recall that a.s. each cluster $\mathcal{C}_{1,j}$ is finite, and $\alpha(\mathcal{C}_{1,j})$ and $\omega(\mathcal{C}_{1,j})$ are respectively, its start- and end-points. Due to property (2.3) we have that $\alpha(\mathcal{C}_{1,j+1}) - \omega(\mathcal{C}_{1,j}) \geq L$, and since we assumed $\chi(\gamma) = 0$, also $\alpha(\mathcal{C}_{1,1}) \geq L$. We set $\tilde{\alpha}_{0,0}^1 = 0, \tilde{\omega}_{0,0}^1 = 1$,

$$\tilde{b}_1 = \left\lfloor \frac{\alpha(\mathcal{C}_{1,1})}{L/3} \right\rfloor, \quad \tilde{b}_j = \left\lfloor \frac{\alpha(\mathcal{C}_{1,j}) - \omega(\mathcal{C}_{1,j-1})}{L/3} \right\rfloor \quad \text{for } j \geq 2, \quad (3.1)$$

($\lfloor \cdot \rfloor$ denotes the integer part; in particular $\tilde{b}_j \geq 3$), while

$$b_j = \begin{cases} \tilde{b}_j, & \text{if } m(\mathcal{C}_{1,j}) = 1; \\ \tilde{b}_j + 1, & \text{if } m(\mathcal{C}_{1,j}) > 1, \end{cases}$$

for $j \geq 1$. Also

$$\tilde{\omega}_{j,0}^1 = \begin{cases} \omega(\mathcal{C}_{1,j}) + 3, & \text{if } m(\mathcal{C}_{1,j}) = 1; \\ \omega(\mathcal{C}_{1,j}), & \text{if } m(\mathcal{C}_{1,j}) > 1, \end{cases} \quad (3.2)$$

for $j \geq 1$.

Now, if $j \geq 0, 1 \leq i \leq \tilde{b}_{j+1} - 1$, we set

$$\tilde{\omega}_{j,i}^1 = \begin{cases} iL/3, & \text{if } j = 0; \\ \omega(\mathcal{C}_{1,j}) + iL/3, & \text{if } j \geq 1. \end{cases} \quad (3.3)$$

Note that this definition holds for $i \leq \tilde{b}_{j+1} - 1$ only and not necessarily for $i = b_{j+1} - 1$. In fact, if $j \geq 0$ and $m(\mathcal{C}_{1,j+1}) > 1$ we make the special definition

$$\tilde{\omega}_{j,b_{j+1}-1}^1 = \alpha(\mathcal{C}_{1,j+1}) - 1. \quad (3.4)$$

For $j \geq 1$, we also define:

$$\tilde{\alpha}_{j,0}^1 = \begin{cases} \tilde{\omega}_{j-1,b_{j-1}}^1 + 1, & \text{if } m(\mathcal{C}_{1,j}) = 1, \\ \alpha(\mathcal{C}_{1,j}), & \text{if } m(\mathcal{C}_{1,j}) > 1. \end{cases} \quad (3.5)$$

Finally, let

$$\tilde{\alpha}_{j,i}^1 = \tilde{\omega}_{j,i-1}^1 + 1, \quad \text{for } j \geq 0, 1 \leq i \leq b_{j+1} - 1. \quad (3.6)$$

For consistency with the notation to be introduced in the next step we write b_j^1 and \tilde{b}_j^1 for b_j and \tilde{b}_j , respectively. We also write $\tilde{\mathcal{C}}_{t,j}, j = 1, 2, \dots$ for the clusters in \mathbf{C}_t of mass at least t , in increasing order. For $t = 1, \tilde{\mathcal{C}}_{1,j} = \mathcal{C}_{1,j}$.

From the facts that $L \geq 12$, property (2.3) for $k = 1$, and $\alpha(\mathcal{C}_{1,1}) \geq L$, we see that the following properties hold for $t = 1$

$$\tilde{\alpha}_{j,i}^t \leq \tilde{\omega}_{j,i}^t \text{ for } j \geq 0, 0 \leq i \leq b_{j+1}^t - 1; \quad (3.7)$$

$$\begin{aligned} &\text{the intervals } [\tilde{\alpha}_{j,i}^t, \tilde{\omega}_{j,i}^t], j \geq 0, 0 \leq i \leq b_{j+1}^t - 1 \text{ (in the order } (j, i) = (0, 0), \dots, \\ &(0, b_1^t - 1), (1, 0), \dots, (1, b_2^t - 1), (2, 0), \dots) \text{ form a partition of } \mathbb{Z}_+; \end{aligned} \quad (3.8)$$

$$\text{each } [\tilde{\alpha}_{j,i}^t, \tilde{\omega}_{j,i}^t] \text{ is a union of intervals } [\tilde{\alpha}_{u,v}^{t-1}, \tilde{\omega}_{u,v}^{t-1}] \quad (3.9)$$

over a finite number of suitable pairs (u, v)

(for $t = 1$ this condition is taken to be fulfilled by convention);

$$\tilde{\mathcal{C}}_{t,j} \subset [\tilde{\alpha}_{j,0}^t, \tilde{\omega}_{j,0}^t], \text{ for each } j \geq 1; \quad (3.10)$$

$$\text{span}(\tilde{\mathcal{C}}_{t,j}) = [\tilde{\alpha}_{j,0}^t, \tilde{\omega}_{j,0}^t] \text{ when } m(\tilde{\mathcal{C}}_{t,j}) > t, \text{ for each } j \geq 1. \quad (3.11)$$

As we shall see, suitable analogues of these properties will hold at all steps in the recursive construction. For $j \geq 0$ and $0 \leq i \leq b_{j+1} - 1$ define the 1-layers

$$\tilde{H}_{j,i}^1 = \bigcup_{s \in [\tilde{\alpha}_{j,i}^1, \tilde{\omega}_{j,i}^1]} H_s^0. \quad (3.12)$$

In particular, from the initial convention, $\tilde{H}_{0,0}^1 = H_0^0 \cup H_1^0$.

Notice that the 1-layers $\tilde{H}_{j,i}^1$, $j \geq 0, 1 \leq i \leq b_{j+1} - 1$, are contained in

$$\bigcup_{\omega(\mathcal{C}_{1,j})+1 \leq s \leq \alpha(\mathcal{C}_{1,j+1})-1} H_s^0$$

and therefore do not contain any bad 0-layers. They will be called *good* 1-layers of *type 2*. For these values of i, j we set

$$\mathcal{D}_{j,i}^1 = \{\tilde{\omega}_{j,i}^1 - 1, \tilde{\omega}_{j,i}^1\} \quad \text{and} \quad \mathcal{D}_{j,i}^{1,\mathcal{K}} = \{\tilde{\omega}_{j,i}^1\}. \quad (3.13)$$

If a layer $\tilde{H}_{j,i}^1$ does not contain a bad line we simply write $\mathcal{K}_{j,i}^1 = [\tilde{\alpha}_{j,i}^1, \tilde{\omega}_{j,i}^1]$. This applies to each $\tilde{H}_{j,i}^1$ with $i \geq 1$. On the other hand, if $j \geq 1$, each of the layers $\tilde{H}_{j,0}^1$ contains exactly $m(\tilde{\mathcal{C}}_{1,j})$ bad 0-layers. When $m(\tilde{\mathcal{C}}_{1,j}) = 1$, (i.e. $\ell(\tilde{\mathcal{C}}_{1,j}) = 0$), $\tilde{H}_{j,0}^1$ is still said to be a *good* 1-layer (of *type 1*, in this case). We then set

$$\begin{aligned} \mathcal{D}_{j,0}^1 &= \{\tilde{\omega}_{j,0}^1 - 1, \tilde{\omega}_{j,0}^1\}, \quad \mathcal{D}_{j,0}^{1,\mathcal{K}} = \{\alpha(\tilde{\mathcal{C}}_{1,j}) - 1\}, \\ \text{and } \mathcal{K}_{j,0}^1 &= [\omega_{j-1,b_j-1}^1 + 1, \alpha(\tilde{\mathcal{C}}_{1,j}) - 1]. \end{aligned} \quad (3.14)$$

If $m(\tilde{\mathcal{C}}_{1,j}) > 1$, then $\tilde{H}_{j,0}^1$ is called a *bad* 1-layer.

Remark. The layer $\tilde{H}_{0,0}^1$ is exceptional. It has not been classified as good or bad above, though there is no harm in calling it good.

The family $\{\tilde{H}_{j,i}^1, j \geq 0, 0 \leq i \leq b_{j+1} - 1\}$ forms a partition of $\tilde{\mathbb{Z}}_+^2$ into horizontal layers. We relabel these layers in increasing order (upwards) as $\mathbf{H}^1 = \{H_j^1\}_{j \geq 1}$. In particular, we see that $H_1^1 = \tilde{H}_{0,0}^1$. When the layer H_j^1 is good, the associated \mathcal{D} , \mathcal{D}^K and \mathcal{K} defined above, also carry the superscript 1 and subscript j .

Now define α_j^1 and ω_j^1 by the requirement

$$H_j^1 = \bigcup_{s \in [\alpha_j^1, \omega_j^1]} H_s^0, \quad (3.15)$$

and write $\mathcal{H}_j^1 = [\alpha_j^1, \omega_j^1]$, and $\mathcal{F}_j^1 = \{\alpha_j^1\}$ for each $j \geq 1$. When H_j^1 does not contain a bad line we have $\mathcal{H}_j^1 = \mathcal{K}_j^1$, as follows from the definition given above.

Remark. The intervals $\mathcal{H}_j^1, j \geq 1$, are simply the intervals $[\tilde{\alpha}_{j,i}^1, \tilde{\omega}_{j,i}^1]$ of (3.8) simply relabeled in upward order. (From the construction $\mathcal{H}_1^1 = \{0, 1\}$.)

Step k. Let $k \geq 2$. Recall that $\tilde{\mathcal{C}}_{t,j}, j = 1, 2, \dots$ are just the clusters in \mathbf{C}_t of mass at least t labeled in increasing space order. Note that $\{\tilde{\mathcal{C}}_{t,j} : j \geq 1\}$ is only a subset of \mathbf{C}_t . Assume now that we have carried out the preceding construction through step $k - 1$ in such a way that (3.7)-(3.11) hold for $1 \leq t \leq k - 1$ with

$$\tilde{b}_1^t = \left\lfloor \frac{\alpha(\tilde{\mathcal{C}}_{t,1})}{L^t/3} \right\rfloor, \quad \tilde{b}_j^t = \left\lfloor \frac{\alpha(\tilde{\mathcal{C}}_{t,j}) - \omega(\tilde{\mathcal{C}}_{t,j-1})}{L^t/3} \right\rfloor \quad \text{for } j \geq 2 \quad (3.16)$$

(in particular $\tilde{b}_j^t \geq 3$ due to our assumption $\chi(\gamma) = 0$ and property (2.3)), while

$$b_j^t = \begin{cases} \tilde{b}_j^t, & \text{if } m(\tilde{\mathcal{C}}_{t,j}) = t, \\ \tilde{b}_j^t + 1, & \text{if } m(\tilde{\mathcal{C}}_{t,j}) > t, \end{cases} \quad (3.17)$$

for $j \geq 1$.

As in (3.12) and (3.15) we can form successively for $2 \leq t \leq k - 1$ the t -layers

$$\tilde{H}_{j,i}^t = \bigcup_{s: \mathcal{H}_s^{t-1} \subset [\tilde{\alpha}_{j,i}^t, \tilde{\omega}_{j,i}^t]} H_s^{t-1} \quad (3.18)$$

and then relabel these layers in increasing upward order as $H_j^t, j \geq 1$. The partition \mathbf{H}^t is then $\{H_j^t\}_{j \geq 1}$. Again following the case $t = 1$ we define α_j^t and ω_j^t by the requirement

$$H_j^t = \bigcup_{s: \mathcal{H}_s^{t-1} \subset [\alpha_j^t, \omega_j^t]} H_s^{t-1}. \quad (3.19)$$

Finally we take

$$\mathcal{H}_j^t = [\alpha_j^t, \omega_j^t]. \quad (3.20)$$

We now consider a cluster $\tilde{\mathcal{C}}_{k,j}$ which lies in \mathbf{C}_k and has mass at least equal to k . If $\ell(\tilde{\mathcal{C}}_{k,j}) \leq k - 1$ (this occurs for instance if $m(\tilde{\mathcal{C}}_{k,j}) = k$, by (2.13)), then even $\tilde{\mathcal{C}}_{k,j} \in \mathbf{C}_{k-1}$, by virtue of (2.16) with $s = t = k - 1$. Hence, by (3.11) for $t = k - 1$ (recall that we are

assuming that this is valid), $\text{span}(\tilde{\mathcal{C}}_{k,j}) = [\tilde{\alpha}_{u,0}^{k-1}, \tilde{\omega}_{u,0}^{k-1}]$ for some u . In particular, there exists an index $i_j \geq 1$ such that

$$\alpha(\tilde{\mathcal{C}}_{k,j}) = \alpha_{i_j}^{k-1} \text{ and } \omega(\tilde{\mathcal{C}}_{k,j}) = \omega_{i_j}^{k-1}.$$

On the other hand, if $\ell(\tilde{\mathcal{C}}_{k,j}) = k$, then $\tilde{\mathcal{C}}_{k,j}$ is made up from constituents $\tilde{\mathcal{C}}_{k-1,s_1}, \tilde{\mathcal{C}}_{k-1,s_2}, \dots, \tilde{\mathcal{C}}_{k-1,s_n}$ with $n \geq 2$, say. These constituents have mass at least k . By applying (3.11) (with t replaced by $k-1$) to $\tilde{\mathcal{C}}_{k-1,s_1}$ and to $\tilde{\mathcal{C}}_{k-1,s_n}$ we see that for each j there exist indices $i_j \leq i'_j$ so that

$$\alpha(\tilde{\mathcal{C}}_{k,j}) = \alpha_{i_j}^{k-1} \text{ and } \omega(\tilde{\mathcal{C}}_{k,j}) = \omega_{i'_j}^{k-1}. \quad (3.21)$$

(For $k = 1$ this is (3.11)). Thus in both cases (3.21) holds, but $i_j = i'_j$ if and only if $\ell(\tilde{\mathcal{C}}_{k,j}) \leq k-1$.

Remark. The subscripts i_j, i'_j above are the labels of the first and last constituents of $\tilde{\mathcal{C}}_{k,j}$ in the enumeration of \mathbf{H}^{k-1} (not of $\mathbf{C}_{k-1}!$), as recursively constructed. They depend on k but we omit this in the (already heavy) notation.

We make the convention that $\tilde{\alpha}_{0,0}^k = 0$ and $\tilde{\omega}_{0,0}^k = \omega_3^{k-1}$. We further make the definitions (3.16) and (3.17) for $t = k$ as well. We also set, for $j \geq 1$:

$$\tilde{\omega}_{j,0}^k = \begin{cases} \omega_{i_j+3}^{k-1}, & \text{if } m(\tilde{\mathcal{C}}_{k,j}) = k; \\ \omega_{i'_j}^{k-1}, & \text{if } m(\tilde{\mathcal{C}}_{k,j}) > k. \end{cases} \quad (3.22)$$

To proceed with the definitions of $\tilde{\alpha}_{j,i}^k$ and $\tilde{\omega}_{j,i}^k$ we need new quantities $s(j, i) = s^k(j, i)$. We take $s(j, i) = s^k(j, i)$ to be the unique value of s for which

$$\begin{aligned} iL^k/3 &\in \mathcal{H}_s^{k-1} \text{ if } j = 0, \\ \omega(\tilde{\mathcal{C}}_{k,j}) + iL^k/3 &\in \mathcal{H}_s^{k-1} \text{ if } j \geq 1. \end{aligned} \quad (3.23)$$

($s(j, i)$ is unique because the intervals in (3.8) form a partition of \mathbb{Z}_+ .) Now, letting \tilde{b}_j^k and b_j^k for $j \geq 1$ be defined by (3.16) and (3.17) with $t = k$, we may set, for $j \geq 0$ and $1 \leq i \leq \tilde{b}_{j+1}^k - 1$:

$$\tilde{\omega}_{j,i}^k = \omega_{s(j,i)}^{k-1}. \quad (3.24)$$

Still for $j \geq 0$ and $1 \leq i \leq \tilde{b}_{j+1}^k - 1$ we set

$$\mathcal{D}_{j,i}^k = [\alpha_{s(j,i)-1}^{k-1}, \omega_{s(j,i)}^{k-1}] \text{ and } \mathcal{D}_{j,i}^{k,\mathcal{K}} = [\alpha_{s(j,i)}^{k-1}, \omega_{s(j,i)}^{k-1}]. \quad (3.25)$$

These sets are listed in the same (lexicographic) order for the pairs (j, i) as in (3.8); $\mathcal{D}_v^{k,\mathcal{K}}$ will be the v -th set in this enumeration of the family $\{\mathcal{D}_{j,i}^{k,\mathcal{K}}\}$. This comment also applies to the family $\{\mathcal{D}_{j,i}^{1,\mathcal{K}}\}$ introduced in (3.13)-(3.14).

If $j \geq 0$ and $m(\tilde{\mathcal{C}}_{k,j+1}) > k$ we set

$$\tilde{\omega}_{j,b_{j+1}^k-1}^k = \alpha(\tilde{\mathcal{C}}_{k,j+1}) - 1. \quad (3.26)$$

We may now define for $j \geq 1$:

$$\tilde{\alpha}_{j,0}^k = \begin{cases} \tilde{\omega}_{j-1,b_j^k-1}^k + 1, & \text{if } m(\tilde{\mathcal{C}}_{k,j}) = k; \\ \alpha_{i_j}^{k-1}, & \text{if } m(\tilde{\mathcal{C}}_{k,j}) > k, \end{cases} \quad (3.27)$$

and, in the case $m(\tilde{\mathcal{C}}_{k,j}) = k$,

$$\mathcal{D}_{j,0}^k = [\alpha_{i_j+2}^{k-1}, \omega_{i_j+3}^{k-1}], \text{ and } \mathcal{D}_{j,0}^{k,\mathcal{K}} = [\alpha_{i_j-1}^{k-1}, \omega_{i_j-1}^{k-1}]. \quad (3.28)$$

Finally, if $j \geq 0$, we set

$$\tilde{\alpha}_{j,i}^k = \tilde{\omega}_{j,i-1}^k + 1 \quad \text{for } 1 \leq i \leq b_{j+1}^k - 1. \quad (3.29)$$

We then set, if $j \geq 0$, $0 \leq i \leq b_{j+1}^k - 1$:

$$\tilde{H}_{j,i}^k = \bigcup_{s: \mathcal{H}_s^{k-1} \subset [\tilde{\alpha}_{j,i}^k, \tilde{\omega}_{j,i}^k]} H_s^{k-1}.$$

The k -layers $\tilde{H}_{j,i}^k$ with $j \geq 0$ and $1 \leq i \leq b_{j+1}^k - 1$ are said to be *good* of *type 2*. The layers $\tilde{H}_{j,0}^k$, with a $j \geq 1$ for which $m(\tilde{\mathcal{C}}_{k,j}) = k$ are also called good (in this case of *type 1*), and we set

$$\mathcal{K}_{j,0}^k = [\tilde{\alpha}_{j,0}^k, \alpha(\tilde{\mathcal{C}}_{k,j}) - 1], \text{ and } \mathcal{K}_{j,i}^k = [\tilde{\alpha}_{j,i}^k, \tilde{\omega}_{j,i}^k]. \quad (3.30)$$

The k -layers $\tilde{H}_{j,0}^k$, with $j \geq 1$ such that $m(\tilde{\mathcal{C}}_{k,j}) > k$ are called *bad*. The layer $\tilde{H}_{0,0}^k$ is again exceptional. The *support* of the layer $\tilde{H}_{j,i}^t$ is the interval $[\tilde{\alpha}_{j,i}^t, \tilde{\omega}_{j,i}^t]$, sometimes written as $\text{supp}(\tilde{H}_{j,i}^t)$.

We shall prove by induction that (3.7)-(3.11) hold for all $t \geq 1$. For this we make the convention that for $r \geq 0$, $\alpha_r^0 = \omega_r^0 = r$ and $\mathcal{H}_r^0 = [r, r] = \{r\}$. We can then define $s^1(j, i)$ in the same way as $s^k(j, i)$ in (3.23) with $k-1$ replaced by 0. We shall also need the following quantities:

$$\begin{aligned} A_t &= \max\{(\omega_j^t - \alpha_j^t) : j \geq 2, H_j^t \text{ is a good } t\text{-layer}\} \\ &= \max\{(\tilde{\omega}_{j,i}^t - \tilde{\alpha}_{j,i}^t) : j \geq 0, 0 \leq i \leq b_{j+1}^t - 1, (j, i) \neq (0, 0), \tilde{H}_{j,i}^t \text{ is a good } t\text{-layer}\} \end{aligned}$$

and

$$\begin{aligned} a_t &= \min\{(\omega_j^t - \alpha_j^t) : j \geq 2, H_j^t \text{ is a good } t\text{-layer}\} \\ &= \min\{(\tilde{\omega}_{j,i}^t - \tilde{\alpha}_{j,i}^t) : j \geq 0, 0 \leq i \leq b_{j+1}^t - 1, (j, i) \neq (0, 0), \tilde{H}_{j,i}^t \text{ is a good } t\text{-layer}\}. \end{aligned}$$

Lemma 3.1. *Let $L \geq 108$ and let γ be an environment with $\chi(\gamma) = 0$. Then the following properties hold for all $t \geq 1$:*

$$\begin{aligned} &\text{The support of a bad } (t-1)\text{-layer cannot intersect the interval} \\ &[1, \alpha(\tilde{\mathcal{C}}_{t,1}) - 1] \text{ or any interval } [\omega(\tilde{\mathcal{C}}_{t,j}) + 1, \alpha(\tilde{\mathcal{C}}_{t,j+1}) - 1], \quad j \geq 1; \end{aligned} \quad (3.31)$$

$$\text{For all } j \geq 0, 1 \leq i \leq b_{j+1}^t - 1, H_{s^t(j,i)}^{t-1} \text{ is a good } (t-1)\text{-layer}; \quad (3.32)$$

If $j \geq 1$ and $m(\tilde{\mathcal{C}}_{t,j}) = t$, then $H_{i_j+\ell}^{t-1}$ is a good $(t-1)$ -layer for $1 \leq \ell \leq L/3$, with i_j as defined after (3.20), and $\tilde{\alpha}_{j,1}^t \leq \tilde{\omega}_{j,1}^t$; (3.33)

$$L^t/4 \leq a_t \leq A_t \leq 2L^t. \quad (3.34)$$

Moreover, the properties (3.7)-(3.11) hold for all $t \geq 1$.

Proof. Note that it follows directly from the definitions that if $\tilde{H}_{j,i}^t$ is a bad t -layer for some $t \geq 1$, then it must be the case that $i = 0$, $m(\tilde{\mathcal{C}}_{t,j}) > t$ and $\text{span}(\tilde{\mathcal{C}}_{t,j}) = [\tilde{\alpha}_{j,0}^t, \tilde{\omega}_{j,0}^t]$.

For $t = 1$, (3.31) is obvious, since each bad line belongs to some cluster $\tilde{\mathcal{C}}_{1,j}$.

Now assume that (3.31) is false for some $t \geq 2$ and that $\tilde{H}_{p,0}^{t-1}$ is a bad $(t-1)$ -layer whose support $[\tilde{\alpha}_{p,0}^{t-1}, \tilde{\omega}_{p,0}^{t-1}]$ intersects $[\omega(\tilde{\mathcal{C}}_{t,j}) + 1, \alpha(\tilde{\mathcal{C}}_{t,j+1})]$ in a point x . (For $j = 0$ we interpret $\omega(\tilde{\mathcal{C}}_{t,0})$ as 0, but for simplicity we restrict ourselves to $j \geq 1$ in the proof of (3.31).) Then, as just observed, $[\tilde{\alpha}_{p,0}^{t-1}, \tilde{\omega}_{p,0}^{t-1}] = \text{span}(\tilde{\mathcal{C}}_{t-1,p})$ for some cluster $\tilde{\mathcal{C}}_{t-1,p}$ of mass at least t . When \mathbf{C}_t is formed from \mathbf{C}_{t-1} , then $\tilde{\mathcal{C}}_{t-1,p}$ is either in the same maximal t -run as the constituents of $\tilde{\mathcal{C}}_{t,j}$, or in the same maximal t -run as the constituents of $\tilde{\mathcal{C}}_{t,j+1}$, or $\tilde{\mathcal{C}}_{t-1,p}$ is disjoint from both these maximal t -runs. In fact only the last situation is possible, because the point x of $\tilde{\mathcal{C}}_{t-1,p}$ lies outside $\text{span}(\tilde{\mathcal{C}}_{t,j}) \cup \text{span}(\tilde{\mathcal{C}}_{t,j+1})$ and $\tilde{\mathcal{C}}_{t-1,p} \in \mathbf{C}_{t-1,t}$. It then follows that $\text{span}(\tilde{\mathcal{C}}_{t,j})$, $\text{span}(\tilde{\mathcal{C}}_{t,j+1})$ and $\text{span}(\tilde{\mathcal{C}}_{t-1,p})$ are (pairwise) disjoint (see (2.17)). But this too is impossible, because $\tilde{\mathcal{C}}_{t,j}$ and $\tilde{\mathcal{C}}_{t,j+1}$ are successive clusters of \mathbf{C}_t of mass $\geq t$. Thus $\tilde{\mathcal{C}}_{t-1,p}$ cannot lie between $\tilde{\mathcal{C}}_{t,j}$ and $\tilde{\mathcal{C}}_{t,j+1}$. Thus, (3.31) holds.

Next we prove (3.32). Assume, to derive a contradiction, that for some $t \geq 1, j \geq 0, 1 \leq i \leq \tilde{b}_{j+1}^t - 1$, $H_{s^t(j,i)}^{t-1}$ is a bad $(t-1)$ -layer. For the purpose of this proof use the abbreviation $r = s^t(j, i)$. By definition of $s^t(j, i)$ we then have

$$\alpha_r^{t-1} \leq \omega(\tilde{\mathcal{C}}_{t,j}) + i \frac{L^t}{3} \leq \omega_r^{t-1} \quad (3.35)$$

if $j \geq 1$, and $\alpha_r^{t-1} \leq i L^t/3 \leq \omega_r^{t-1}$ if $j = 0$. For the sake of argument we assume that $j \geq 1$; the case $j = 0$ is similar. But for $1 \leq i \leq \tilde{b}_{j+1}^t - 1$, it holds (by (3.16))

$$\omega(\tilde{\mathcal{C}}_{t,j}) + 1 \leq \omega(\tilde{\mathcal{C}}_{t,j}) + i \frac{L^t}{3} \leq \alpha(\tilde{\mathcal{C}}_{t,j+1}) - \frac{L^t}{3}.$$

In other words,

$$\omega(\tilde{\mathcal{C}}_{t,j}) + i \frac{L^t}{3} \in [\omega(\tilde{\mathcal{C}}_{t,j}) + 1, \alpha(\tilde{\mathcal{C}}_{t,j+1}) - 1] \cap [\alpha_r^{t-1}, \omega_r^{t-1}].$$

By virtue of (3.31) this contradicts our assumption that $[\alpha_r^{t-1}, \omega_r^{t-1}]$ is a bad $(t-1)$ -layer. Thus (3.32) must hold.

The proof of (3.33) has several similarities with that of (3.32). For this property as well as for (3.7)-(3.11) we use a proof by induction. First, the first part of (3.33) for $t = 1$ is easy. Indeed, if $m(\mathcal{C}_{1,j}) = 1$, then $(\mathcal{C}_{1,j})$ is a singleton and $i_j = i'_j$ is such that $\alpha(\tilde{\mathcal{C}}_{1,j}) = \omega(\tilde{\mathcal{C}}_{1,j}) =$

$\alpha_{i_j}^0 = \omega_{i_j}^0 = i_j$. The layers $H_{i_j+\ell}^0, 1 \leq \ell \leq L/3$, therefore consist of the lines H_p , with $p \in [\omega(\tilde{\mathcal{C}}_{1,j}) + 1, \omega(\tilde{\mathcal{C}}_{1,j}) + L/3] \subset [\omega(\tilde{\mathcal{C}}_{1,j}) + 1, \alpha(\tilde{\mathcal{C}}_{1,j+1}) - 1]$, because $\alpha(\tilde{\mathcal{C}}_{1,j+1}) - \omega(\tilde{\mathcal{C}}_{1,j}) \geq L$, by virtue of (2.3). It then follows from (3.31) that all the layers $H_{i_j+\ell}^0, 1 \leq \ell \leq L/3$ must be good 0-layers, so that the first part of (3.33) holds for $t = 1$.

As for the last part of (3.33), if $m(\tilde{\mathcal{C}}_{1,j}) = 1$ and $t = 1$, this is equivalent (by (3.6),(3.2) and (3.3)) to

$$\tilde{\omega}_{j,0}^1 + 1 = \omega(\tilde{\mathcal{C}}_{1,j}) + 4 \leq \omega(\tilde{\mathcal{C}}_{1,j}) + \frac{L}{3}.$$

Clearly this last inequality holds when $L \geq 12$, so that (3.33) holds for $t = 1$. One can also check by hand that the properties (3.34) and (3.7)-(3.11) hold for $t = 1$. E.g., for (3.33) use (3.6) and (3.3). We therefore only have to verify the induction step for (3.33),(3.34) and (3.7)-(3.11).

Assume then that (3.33), (3.34) and (3.8) have already been proven for $t = k - 1 \geq 1$. It then follows from (3.8) that

$$\alpha_{u+1}^{k-1} = \omega_u^{k-1} + 1, u \geq 0. \quad (3.36)$$

We can therefore write

$$\omega_{i_j+\ell}^{k-1} = \omega_{i_j}^{k-1} + \sum_{v=1}^{\ell} [\omega_{i_j+v}^{k-1} - \alpha_{i_j+v}^{k-1} + 1].$$

Now assume, to derive a contradiction, that $m(\tilde{\mathcal{C}}_{k,j}) = k$ and that $H_{i_j+\ell}^{k-1}$ is a bad $(k-1)$ -layer for some $1 \leq \ell \leq L/3$. Pick $\ell \in [1, L/3]$ to be minimal with this property. Then $H_{i_j+v}^{k-1}$ is a good $(k-1)$ -layer for $1 \leq v < \ell$, so that

$$\omega_{i_j+v}^{k-1} - \alpha_{i_j+v}^{k-1} \leq A_{k-1} \leq 2L^{k-1}, 1 \leq v < \ell,$$

(by (3.34) for $t = k - 1$). In particular,

$$\alpha_{i_j+\ell}^{k-1} - \omega_{i_j}^{k-1} = \omega_{i_j+\ell-1}^{k-1} + 1 - \omega_{i_j}^{k-1} \leq (\ell - 1)[2L^{k-1} + 1] + 1 \leq 3\ell L^{k-1} - 1. \quad (3.37)$$

But α_u^{k-1} is increasing in u by (3.8) for $t = k - 1$. Therefore,

$$\begin{aligned} \omega(\tilde{\mathcal{C}}_{k,j}) &= \omega_{i'_j}^{k-1} = \omega_{i_j}^{k-1} \text{ (because we took } m(\tilde{\mathcal{C}}_{k,j}) = k) \\ &= \alpha_{i_j+1}^{k-1} - 1 \text{ (by (3.36))} < \alpha_{i_j+\ell}^{k-1} \leq \omega(\tilde{\mathcal{C}}_{k,j}) + 3\ell L^{k-1} - 1 < \alpha(\tilde{\mathcal{C}}_{k,j+1}), \end{aligned} \quad (3.38)$$

where the last inequality follows from $\alpha(\tilde{\mathcal{C}}_{k,j+1}) - \omega(\tilde{\mathcal{C}}_{k,j}) = d(\tilde{\mathcal{C}}_{k,j}, \tilde{\mathcal{C}}_{k,j+1}) \geq L^k$, which in turn follows from (2.3). Thus

$$\alpha_{i_j+\ell}^{k-1} \in [\omega(\tilde{\mathcal{C}}_{k,j}) + 1, \alpha(\tilde{\mathcal{C}}_{k,j+1}) - 1] \cap [\alpha_{i_j+\ell}^{k-1}, \omega_{i_j+\ell}^{k-1}]$$

and we have again arrived at a contradiction with (3.31). This proves that $H_{i_j+\ell}^{k-1}$ is a good $(k-1)$ -layer for $1 \leq \ell \leq L/3$, which is the first claim in (3.33) with $t = k$.

Now, to prove the last part of (3.33) with $t = k$, note that under the condition $m(\tilde{\mathcal{C}}_{k,j}) = k$ it is equivalent to

$$\omega_{j,0}^k + 1 = \omega_{i_j+3}^{k-1} + 1 \leq \omega_{s^k(j,1)}^{k-1}, \quad (3.39)$$

by virtue of (3.29), (3.22) and (3.24). But, just as in (3.37),

$$\omega_{i_j+3}^{k-1} \leq \omega_{i_j}^{k-1} + 3(2L^{k-1} + 1) = \omega(\tilde{\mathcal{C}}_{k,j}) + 6L^{k-1} + 3.$$

On the other hand, by the definition of $s(j, 1)$ it holds

$$\omega_{s^k(j,1)}^{k-1} \geq \omega(\tilde{\mathcal{C}}_{k,j}) + \frac{L^k}{3}.$$

Thus, (3.39) does indeed hold (for $L \geq 108$) and (3.33) for $t = k$ follows.

Next we turn to (3.34) for $t = k$. We only give some representative parts of the argument, and leave the remaining parts to the reader. In particular, for the last inequality we only estimate

$$\max\{(\tilde{\omega}_{j,i}^k - \tilde{\alpha}_{j,i}^k) : j \geq 1, m(\tilde{\mathcal{C}}_{k,j}) > k, 0 \leq i \leq b_{j+1}^k - 1, (j, i) \neq (0, 0), \tilde{\mathcal{H}}_{j,i}^k \text{ is a good } k\text{-layer}\}.$$

Note that the case with $j \geq 1, i = 0$ does not have to be considered, because $\tilde{H}_{j,0}^k$ is bad when $j \geq 1, m(\tilde{\mathcal{C}}_{k,j}) > k$. We now separately consider the cases

- (i) $j \geq 1, 2 \leq i \leq \tilde{b}_{j+1}^k - 1$;
- (ii) $j \geq 1, i = 1$;
- (iii) $j \geq 1, \tilde{b}_{j+1}^k \leq i \leq b_{j+1}^k - 1$.

For (j, i) in case (i) we have

$$\begin{aligned} \tilde{\omega}_{j,i}^k - \tilde{\alpha}_{j,i}^k &= \omega_{s^k(j,i)}^{k-1} - \tilde{\omega}_{j,i-1}^k - 1 = \omega_{s^k(j,i)}^{k-1} - \omega_{s^k(j,i-1)}^{k-1} - 1 \\ &\leq \omega_{s^k(j,i)}^{k-1} - [\omega(\tilde{\mathcal{C}}_{k,j}) + i\frac{L^k}{3}] + [\omega(\tilde{\mathcal{C}}_{k,j}) + i\frac{L^k}{3}] - [\omega(\tilde{\mathcal{C}}_{k,j}) + (i-1)\frac{L^k}{3}] \\ &\quad (\text{compare with the right hand inequality in (3.35)}) \\ &\leq A_{k-1} + \frac{L^k}{3} \end{aligned}$$

(note that this step uses (3.32) for $t = k$ and the induction hypothesis).

For (j, i) in case (ii) we have similarly

$$\begin{aligned} \tilde{\omega}_{j,i}^k - \tilde{\alpha}_{j,i}^k &= \omega_{s^k(j,1)}^{k-1} - \tilde{\omega}_{j,0}^k - 1 \\ &\leq A_{k-1} + \omega(\tilde{\mathcal{C}}_{k,j}) + \frac{L^k}{3} - \omega_{i'_j}^{k-1} - 1 \\ &= A_{k-1} + \omega(\tilde{\mathcal{C}}_{k,j}) + \frac{L^k}{3} - \omega(\tilde{\mathcal{C}}_{k,j}) - 1. \end{aligned}$$

Finally, case (iii) is non-empty if and only if $m(\tilde{\mathcal{C}}_{k,j+1}) > k$ so that $b_{j+1}^k = \tilde{b}_{j+1}^k + 1$. We then have for $i = b_{j+1}^k - 1$,

$$\begin{aligned} \tilde{\omega}_{j,i}^k - \tilde{\alpha}_{j,i}^k &= \alpha(\tilde{\mathcal{C}}_{k,j+1}) - 1 - \tilde{\omega}_{j,b_{j+1}^k-2}^k - 1 \text{ (by (3.26) and (3.29))} \\ &\leq \alpha(\tilde{\mathcal{C}}_{k,j+1}) - \omega_{s(j,b_{j+1}^k-2)}^{k-1} \text{ (by (3.24))} \leq \alpha(\tilde{\mathcal{C}}_{k,j+1}) - \omega(\tilde{\mathcal{C}}_{k,j}) - [b_{j+1}^k - 2] \frac{L^k}{3} \\ &\leq L^k \text{ (by (3.16)).} \end{aligned}$$

Thus in all cases checked so far we found that

$$\tilde{\omega}_{j,i}^k - \tilde{\alpha}_{j,i}^k \leq A_{k-1} + L^k,$$

and it turns out that this inequality holds in all cases. Together with the induction hypothesis this shows that for $L \geq 12$

$$A_k \leq A_{k-1} + L^k \leq 2L^{k-1} + L^k \leq 2L^k.$$

This is the desired first inequality in (3.34) with $t = k$.

The last case we check is the case of the second inequality of (3.34) when $j = 0$, $i = 1$, and still $m(\tilde{\mathcal{C}}_{k,j}) > k$. In this case we find (using the definition of $s^k(0,1)$):

$$\tilde{\omega}_{0,1}^k - \tilde{\alpha}_{0,1}^k = \omega_{s^k(0,1)}^{k-1} - \tilde{\omega}_{0,0}^k - 1 = \omega_{s^k(0,1)}^{k-1} - \omega_3^{k-1} - 1 \geq \frac{L^k}{3} - \omega_3^{k-1} - 1. \quad (3.40)$$

To use this estimate we need an upper bound for ω_3^{k-1} . Now, by (3.8) for $t = k-1$, $\omega_\ell^{k-1} = \tilde{\omega}_{0,\ell-1}^{k-1}$, for $1 \leq \ell \leq \tilde{b}_1^{k-1}$ (and $\tilde{b}_1^{k-1} \geq 3$). Therefore, $\alpha_\ell^{k-1} = \tilde{\alpha}_{0,\ell-1}^{k-1} = \tilde{\omega}_{0,\ell-2}^{k-1} + 1$ for $\ell = 2$ or 3 (see (3.29)). Thus,

$$\begin{aligned} \omega_3^{k-1} &= \tilde{\omega}_{0,2}^{k-1} = [\tilde{\omega}_{0,2}^{k-1} - \tilde{\alpha}_{0,2}^{k-1}] + \tilde{\omega}_{0,1}^{k-1} + 1 \\ &\leq A_{k-1} + \tilde{\omega}_{0,1}^{k-1} + 1 \text{ (use (3.33))} \leq 2A_{k-1} + \tilde{\omega}_{0,0}^{k-1} + 2 \\ &= 2A_{k-1} + \omega_3^{k-2} + 2 \text{ (by definition).} \end{aligned}$$

Iteration of this inequality shows that

$$\omega_3^{k-1} \leq \sum_{t=1}^{k-1} (2L^t + 2) + 2 \leq 8L^{k-1}, \quad (3.41)$$

by the first part of (3.34) and the choice of ω_3^0 . We note in passing that this last estimate also gives

$$\max\{A_k, (\omega_{0,0}^k - \alpha_{0,0}^k)\} \leq 2L^k, \quad (3.42)$$

by our choice of $\alpha_{0,0}^t, \omega_{0,0}^t$. Substitution of the estimate (3.41) into (3.40) shows that

$$\tilde{\omega}_{j,i}^k - \tilde{\alpha}_{j,i}^k \geq \frac{L^k}{3} - 8L^{k-1} - 1 \geq \frac{L^k}{4},$$

provided $L \geq 108$, $j = 0, i = 1$, and $m(\tilde{\mathcal{C}}_{k,j}) > k$. In fact this inequality remains valid for all good k -layers $\tilde{H}_{j,i}^k$, $0 \leq i \leq b_{j+1}^k - 1, (j, i) \neq (0, 0)$. Thus (3.34) holds also for $t = k$.

Next we must prove (3.7) for $t = k$. For $(j, i) = (0, 0)$ this is immediate from our choice of $\tilde{\alpha}_{0,0}^k$ and $\tilde{\omega}_{0,0}^k$. For $(j, i) = (0, 1)$, (3.7) requires that

$$\tilde{\alpha}_{0,1}^k = \tilde{\omega}_{0,0}^k + 1 = \omega_3^{k-1} + 1 \leq \tilde{\omega}_{0,1}^k \quad (3.43)$$

(see (3.29)). But we already saw in (3.41) that $\omega_3^{k-1} \leq 8L^{k-1}$, while $\tilde{\omega}_{0,1}^k = \omega_{s^k(0,1)}^{k-1}$ (see (3.24)) $\geq L^k/3$ (by definition of $s^k(0, 1)$). Thus (3.43) holds and $\tilde{\alpha}_{0,1}^k \leq \tilde{\omega}_{0,1}^k$ when $L \geq 108$.

We also proved (3.7) when $j \geq 1, i = 1$ and $m(\tilde{\mathcal{C}}_{t,j}) = t$, in (3.33).

The remaining cases of (3.7) are routine. Also (3.8)-(3.11) do not require any new ideas, but only tedious definition pushing. We leave the verification to the reader. \square

Remark. Property (3.31) and the arguments for its proof also show that a t -layer $\tilde{H}_{j,i}^t$ with $j \geq 1$ and which is good of type 2 does not contain any bad $(t-1)$ -layer. On the other hand, if for some $j \geq 1$, $m(\tilde{\mathcal{C}}_{t,j}) = t$, then the t -layer $\tilde{H}_{j,0}^t$ is good of type 1, and contains exactly one bad $(t-1)$ -layer.

Indeed, the support of good t -layers of type 2 are of the form $[\tilde{\alpha}_{j,i}^t, \tilde{\omega}_{j,i}^t]$ with $1 \leq i \leq b_{j+1}^t - 1$ and one can check that these are contained in the interval $[\omega(\tilde{\mathcal{C}}_{t,j} + 1, \alpha(\tilde{\mathcal{C}}_{t,j+1}) - 1]$ which cannot intersect any cluster of mass at least t by (3.31).

If, however, $[\tilde{\alpha}_{j,0}^t, \tilde{\omega}_{j,0}^t]$ corresponds to a good t -layer of type 1, i.e., if $m(\tilde{\mathcal{C}}_{t,j}) = t$, then the formulae (3.27) and (3.22) (see also (3.10)) can be used to show that $[\tilde{\alpha}_{j,0}^t, \tilde{\omega}_{j,0}^t] \supset [\alpha(\tilde{\mathcal{C}}_{t,j}), \omega(\tilde{\mathcal{C}}_{t,j})]$. By (3.21) and its proof we then have $[\alpha(\tilde{\mathcal{C}}_{t,j}), \omega(\tilde{\mathcal{C}}_{t,j})] = [\tilde{\alpha}_{u,0}^{t-1}, \tilde{\omega}_{u,0}^{t-1}]$ for some $u \geq 1$. Moreover, the layer $\tilde{H}_{u,0}^{t-1}$ is a bad $(t-1)$ -layer, because $m(\tilde{\mathcal{C}}_{t,j}) > t-1$. In this case $[\tilde{\alpha}_{u,0}^{t-1}, \tilde{\omega}_{u,0}^{t-1}]$ necessarily is the only bad $(t-1)$ -layer in $[\tilde{\alpha}_{j,0}^t, \tilde{\omega}_{j,0}^t]$. To see this, note that if $[\tilde{\alpha}_{j,0}^t, \tilde{\omega}_{j,0}^t]$ would contain another bad $(t-1)$ -layer, then it would have to contain a cluster $\tilde{\mathcal{C}}_{t,j'}$ from \mathbf{C}_t of mass t , but $j' \neq j$. However, that cluster would be contained in $[\tilde{\alpha}_{j',0}^t, \tilde{\omega}_{j',0}^t]$, and this interval is disjoint from $[\tilde{\alpha}_{j,0}^t, \tilde{\omega}_{j,0}^t]$ (by (3.8)), so that $\tilde{\mathcal{C}}_{t,j'}$ cannot lie in $[\tilde{\alpha}_{j,0}^t, \tilde{\omega}_{j,0}^t]$ after all.

Finally we note that $\tilde{b}_j^k \geq 3$ and (3.8) show that any bad k -layer $\tilde{H}_{j,0}^k$ is followed by the two k -layers $\tilde{H}_{j,i}^k$, $i = 1, 2$, which are good of type 2 (as we had claimed already in the second paragraph of this section).

We have now proven that the family $\{\tilde{H}_{j,i}^k, j \geq 0, 0 \leq i \leq b_{j+1}^k - 1\}$ again gives a partition of $\tilde{\mathbb{Z}}_+^2$ and we relabel it as $\{H_j^k, j \geq 1\}$, in increasing order. (In this notation $\tilde{H}_{0,0}^k = H_1^k$.) As before, when H_j^k is a good layer, the associated \mathcal{D} , \mathcal{D}^K , and \mathcal{K} are written as \mathcal{D}_j^k etc., i.e., they carry the superscript k and subscript j . Now define α_j^k and ω_j^k as in (3.19) with $t = k$, and set $\mathcal{H}_j^k = [\alpha_j^k, \omega_j^k]$, as in (3.20). We set

$$s_j = \min\{s: H_s^{k-1} \subset H_j^k\} \text{ and } \mathcal{F}_j^k = \mathcal{H}_{s_j}^{k-1}.$$

The intervals $\mathcal{H}_j^k, j \geq 1$, give a partition of \mathbb{Z}_+ . (From the construction $\mathcal{H}_1^k = \{0, \dots, \omega_3^{k-1}\}$.)

Remark. Some of the preceding arguments can be used to give a lower bound for $i_{j+1} - i'_j$ as follows. By (2.3)

$$\alpha_{i_{j+1}}^{k-1} - \omega_{i'_j}^{k-1} = \alpha(\tilde{\mathcal{C}}_{k,j+1}) - \omega(\tilde{\mathcal{C}}_{k,j}) \geq L^k \text{ for } j \geq 1.$$

Moreover, by (3.31) and (3.8) with $t = k - 1$, the interval $[\omega(\tilde{\mathcal{C}}_{k,j}) + 1, \alpha(\tilde{\mathcal{C}}_{k,j+1}) - 1]$ is a disjoint union of the supports of a some good layers. More precisely,

$$L^k \leq \alpha(\tilde{\mathcal{C}}_{k,j+1}) - \omega(\tilde{\mathcal{C}}_{k,j}) = \alpha_{i_{j+1}}^{k-1} - \omega_{i'_j}^{k-1} = 1 + \sum [\alpha_u^{k-1} - \omega_u^{k-1} + 1],$$

where the sum over u runs over all u for which $\mathcal{H}_u^{k-1} \subset [\omega(\tilde{\mathcal{C}}_{k,j}) + 1, \alpha(\tilde{\mathcal{C}}_{k,j+1}) - 1]$. There are $i_{j+1} - i'_j$ such summands and each one of them is at most $2L^{k-1} + 1$, by virtue of (3.34). Thus

$$i_{j+1} - i'_j \geq \frac{L^k - 1}{2L^{k-1} + 1} \geq \frac{L}{6} \text{ for each } j \geq 0. \quad (3.44)$$

(i'_0 is to be taken as 0 here; $\chi(\gamma) = 0$ should be used instead of (2.3) to derive this estimate.)

Renormalized sites.

To define renormalized sites we first choose constants c and $L \geq 108$ such that:

$$c < \frac{3}{14}s(p_G), \quad c^{-1} \in \mathbb{N} \text{ and } \frac{1}{2}cL \in \mathbb{N}, \quad (3.45)$$

where $s(p_G)$ is the asymptotic right-edge speed of homogeneous oriented site percolation on $\tilde{\mathbb{Z}}_+^2$ with the probability of a site being open equal to p_G . (see [6] for the definition of right edge speed). We then define the renormalized k -sites $S_{u,v}^k$ as follows:

Step 0. $S_{u,v}^0 = (u, v)$, for $(u, v) \in \tilde{\mathbb{Z}}_+^2$;

Step k. For $k \geq 1$, $(u, v) \in \tilde{\mathbb{Z}}_+^2 \setminus (0, 0)$

$$S_{u,v}^k = \left(\left(\frac{u-1}{2}(cL)^k, \frac{u+1}{2}(cL)^k \right] \times \mathcal{H}_v^k \right) \cap \tilde{\mathbb{Z}}_+^2; \quad (3.46)$$

Remark. It is clear that for fixed $k \geq 1$ the $S_{u,v}^k$ with (u, v) running over $\tilde{\mathbb{Z}}_+^2 \setminus (0, 0)$ form a partition of $\tilde{\mathbb{Z}}_+^2$ (see (3.8)). We further define

$$\mathring{S}_{u,v}^k = \bigcup S_{x,y}^{k-1}$$

with the union running over all (x, y) such that $S_{x,y}^{k-1} \subset S_{u,v}^k$. For $k \geq 2$ and fixed (u, v) it is generally not the case that $\mathring{S}_{u,v}^k = S_{u,v}^k$. Thus $S_{u,v}^k$ is not quite a (disjoint) union of $(k-1)$ -sites. However,

$$\mathring{S}_{u,v}^k \subset S_{u,v}^k \subset \bigcup S_{x,y}^{k-1}, \quad (3.47)$$

where the union in the right hand side is over all (x, y) for which $S_{x,y}^{k-1} \cap S_{u,v}^k \neq \emptyset$. Thus the points of $S_{u,v}^k$ which are not contained in $\mathring{S}_{u,v}^k$ lie within distance $(cL)^{k-1}$ of the vertical

boundary of $S_{u,v}^k$, because the projection of any $(k-1)$ -site on the horizontal axis has diameter at most $(cL)^{k-1}$.

Note that by the definitions (3.18)-(3.20) and $\tilde{\alpha}_{0,0}^k = 0, \tilde{\omega}_{0,0}^k = \omega_3^{k-1}$ one has for $k \geq 2$,

$$H_1^k = \tilde{H}_{0,0}^k = \bigcup_{s: \mathcal{H}_s^{k-1} \subset [\tilde{\alpha}_{0,0}^k, \tilde{\omega}_{0,0}^k]} H_s^{k-1} = H_1^{k-1} \cup H_2^{k-1} \cup H_3^{k-1}.$$

Thus, by iterating this relation we see that for any $k \geq 1$, the layer H_1^k is the union of $H_2^t \cup H_3^t$ for $1 \leq t \leq k-1$ and $H_1^1 = H_0 \cup H_1$.

A renormalized site $S_{u,v}^k$ with $v \geq 2$ is called *good (of type ℓ)* when the k -layer H_v^k is good (of type ℓ). We state two elementary properties of the *good* renormalized sites $S_{u,v}^k$ with $v \geq 2$:

i) the number of horizontal $(k-1)$ -layers intersecting $S_{u,v}^k$ does not depend on u . Any given good k -layer H^k intersects at most $8L$ good $(k-1)$ -layers (by (3.34) and the remark a few lines after (3.43)) and at most one bad $(k-1)$ -layer (by (3.11) and (3.31)). The support of any bad $(k-1)$ -layer which could intersect H^k has diameter at most $3L^{k-1}$ (since H^k being good, its support cannot contain a cluster of mass greater than k , and (2.23) holds). By (3.34) and the remark following (3.43) again, we see that the number of good $(k-1)$ -layers intersecting H^k is at least $(L^k/4 - 3L^{k-1})/(2L^{k-1}) \geq L/9$ (provided $L \geq 108$).

ii) the intersection of $S_{u,v}^k$ with a $(k-1)$ -layer, if not empty, contains exactly cL $(k-1)$ -sites.

Structure of good sites.

As stated above, a good k -site $S_{u,v}^k$ is called good of *type 1* or *type 2* according as the corresponding k -layer H_v^k is good of type 1 or 2 ($v \geq 2$). It follows from the remark after Lemma 3.1 that good k -sites of type 1 are those good k -sites that contain a layer of bad $(k-1)$ -sites. On the other hand, good k -sites of type 2 contain only good $(k-1)$ -sites. If $S_{u,v}^k$ is a good k -site we let

$$D_l(S_{u,v}^k) = \left[\left(\frac{u-1}{2} + \frac{1}{12} \right) (cL)^k, \left(\frac{u}{2} - \frac{1}{3} \right) (cL)^k \right] \times \mathcal{D}_v^k,$$

$$D_r(S_{u,v}^k) = \left[\left(\frac{u}{2} + \frac{1}{3} \right) (cL)^k, \left(\frac{u+1}{2} - \frac{1}{12} \right) (cL)^k \right] \times \mathcal{D}_v^k,$$

as well as

$$D_l^{\mathcal{K}}(S_{u,v}^k) = \left[\left(\frac{u-1}{2} + \frac{1}{12} \right) (cL)^k, \left(\frac{u}{2} - \frac{1}{3} \right) (cL)^k \right] \times \mathcal{D}_v^{k,\mathcal{K}},$$

$$D_r^{\mathcal{K}}(S_{u,v}^k) = \left[\left(\frac{u}{2} + \frac{1}{3} \right) (cL)^k, \left(\frac{u+1}{2} - \frac{1}{12} \right) (cL)^k \right] \times \mathcal{D}_v^{k,\mathcal{K}}, \quad (3.48)$$

where $\mathcal{D}_v^k, \mathcal{D}_v^{k,\mathcal{K}}$ have been defined before (see (3.13), (3.14), (3.25) and the lines following it, and (3.28)).

We remind the reader of (3.30). After rearranging the pairs (j, i) in the order of (3.8) this says for a good layer H_v^k we have

$$\mathcal{K}_v^k = \mathcal{H}_v^k = [\alpha_v^k, \omega_v^k] \text{ if } H_v^k \text{ is good of type 2,} \quad (3.49)$$

and

$$\mathcal{K}_v^k = [\alpha_v^k, \alpha(\tilde{\mathcal{C}}_{k,j}) - 1] \text{ if } H_v^k \text{ is good of type 1 and } \mathcal{H}_v^k = [\tilde{\alpha}_{j,0}^k, \tilde{\omega}_{j,0}^k]. \quad (3.50)$$

We next define

$$\text{Ker}(S_{u,v}^k) = S_{u,v}^k \cap (\mathbb{Z} \times \mathcal{K}_v^k); \quad (3.51)$$

this set is called the *kernel* of $S_{u,v}^k$. Note that $\text{Ker}(S_{u,v}^k)$ equals $S_{u,v}^k$ if this site is good of type 2, but is a strict subset of $S_{u,v}^k$ if $S_{u,v}^k$ is good of type 1. Basically, if $S^k(u, v)$ is a good k -site of type 1, then its projection on the vertical axis contains exactly one cluster $\tilde{\mathcal{C}}_{k,j}$ of mass k (and none of mass greater than k). The kernel of S^k is then the part of $S^k(u, v)$ whose projection on the vertical axis lies below $\tilde{\mathcal{C}}_{k,j}$ (compare (3.49) and (3.50)). We observe that if H_v^k is a good layer and $k \geq 1$, then

$$\text{top line of } \text{Ker}(S_{u,v}^k) = \text{top line of } \mathcal{K}_v^k = \text{top line of } \mathcal{D}_v^{k,\mathcal{K}} = \text{top line of } D_\theta^\mathcal{K}(S_{u,v}^k). \quad (3.52)$$

The second equality here follows from (3.13), (3.14), (3.25), (3.28) and the fact that $\tilde{\omega}_{i_j-1}^{k-1} = \tilde{\alpha}_{i_j}^{k-1} - 1$ (by (3.8) for $t = k - 1$).

We define further

$$\mathcal{F}_v^k = \mathcal{H}_{w_v}^{k-1}, \quad (3.53)$$

where $w_v = \min\{w : H_w^{k-1} \subset H_v^k\}$, and

$$F(S_{u,v}^k) = [(u/2 - 1/6)(cL)^k, (u/2 + 1/6)(cL)^k] \times \mathcal{F}_v^k. \quad (3.54)$$

$F(S_{u,v}^k)$ is, roughly speaking, the middle third of the lowest $(k-1)$ -layer in $S_{u,v}^k$. The $(k-1)$ -sites contained in $F(S_{u,v}^k)$ are said to be *centrally located* in $S_{u,v}^k$.

A reversed partition.

The good k -sites were constructed in a way which ensures high crossing probabilities for open paths from the bottom of the k -site. A technical tool used in the proof requires also good crossing probabilities in the downwards direction. This is achieved by an appropriate modification of the horizontal layers.

Reversed sites. We begin with the definition of the reversed layers.

Step 0. The 0-layers are $\hat{H}_j^0 = H_j$.

Step 1. Take $\mathbf{C}_1 = \{\mathcal{C}_{1,j}\}_{j \geq 1}$ as before. Take \tilde{b}_j as in (3.1), and set

$$\hat{b}_1 = \tilde{b}_1 + 1 \text{ and } \hat{b}_j = \begin{cases} \tilde{b}_j, & \text{if } m(\mathcal{C}_{1,j-1}) = 1 \\ \tilde{b}_j + 1, & \text{if } m(\mathcal{C}_{1,j-1}) > 1 \end{cases} \quad \text{for } j \geq 2. \quad (3.55)$$

With the convention $\widehat{\omega}_{0,0}^1 = 0$ and $\widehat{\alpha}_{0,0}^1 = 1$, we set, for each $j \geq 1$,

$$\widehat{\omega}_{j,0}^1 = \begin{cases} \alpha(\mathcal{C}_{1,j}) - 3, & \text{if } m(\mathcal{C}_{1,j}) = 1, \\ \alpha(\mathcal{C}_{1,j}), & \text{if } m(\mathcal{C}_{1,j}) > 1, \end{cases} \quad (3.56)$$

and

$$\widehat{\alpha}_{j,0}^1 = \begin{cases} \widetilde{\omega}_{j,1}^1, & \text{if } m(\mathcal{C}_{1,j}) = 1, \\ \omega(\mathcal{C}_{1,j}), & \text{if } m(\mathcal{C}_{1,j}) > 1, \end{cases} \quad (3.57)$$

with $\widetilde{\omega}_{j,1}^1$ given by (3.3). If $m(\mathcal{C}_{1,j}) = 1$, then we set, for $j \geq 1$,

$$\widehat{\alpha}_{j,i}^1 = \begin{cases} \widetilde{\omega}_{j,i+1}^1, & \text{if } 1 \leq i \leq \widehat{b}_{j+1} - 2, \\ \widehat{\omega}_{j+1,0}^1 - 1, & \text{if } i = \widehat{b}_{j+1} - 1, \end{cases}$$

while, for $m(\mathcal{C}_{1,j}) > 1$ or $j = 0$ (and $m(\mathcal{C}_{1,0}) = 1$), we set

$$\widehat{\alpha}_{j,i}^1 = \begin{cases} \widetilde{\omega}_{j,i}^1, & \text{if } 1 \leq i \leq \widehat{b}_{j+1} - 2, \\ \widehat{\omega}_{j+1,0}^1 - 1, & \text{if } i = \widehat{b}_{j+1} - 1. \end{cases} \quad (3.58)$$

Irrespective of the value of $m(\mathcal{C}_{1,j})$ we further take for $j \geq 0$

$$\widehat{\omega}_{j,i}^1 = \widehat{\alpha}_{j,i-1}^1 + 1 \quad \text{if } 1 \leq i \leq \widehat{b}_{j+1} - 1. \quad (3.59)$$

For $j \geq 0$ and $0 \leq i \leq \widehat{b}_{j+1} - 1$ define

$$\widehat{H}_{j,i}^1 = \bigcup_{s \in [\widehat{\omega}_{j,i}^1, \widehat{\alpha}_{j,i}^1]} \widehat{H}_s^0.$$

The family $\{\widehat{H}_{j,i}^1, j \geq 0, 0 \leq i \leq \widehat{b}_{j+1} - 1\}$ is relabeled as $\widehat{\mathbf{H}}^1 = \{\widehat{H}_j^1\}_{j \geq 1}$ in increasing order. We then define $\widehat{\omega}_j^1$ and $\widehat{\alpha}_j^1$ through the requirement

$$\widehat{H}_j^1 = \bigcup_{s \in [\widehat{\omega}_j^1, \widehat{\alpha}_j^1]} \widehat{H}_s^0,$$

for each j , and we set $\widehat{\mathcal{H}}_j^1 = [\widehat{\omega}_j^1, \widehat{\alpha}_j^1]$.

Step k. We now consider the clusters in $\mathbf{C}_k = \{\mathcal{C}_{k,j}\}_{j \geq 1}$ which have mass at least $k \geq 2$, and rename them as $\widetilde{\mathcal{C}}_{k,j}, j \geq 1$ (always in increasing order). We make the convention that $\widehat{\omega}_{0,0}^k = 0$ and $\widehat{\alpha}_{0,0}^k = \omega_3^{k-1}$.

$$\widehat{b}_1^k = \widetilde{b}_1^k + 1, \text{ and } \widehat{b}_j^k = \begin{cases} \widetilde{b}_j^k, & \text{if } m(\widetilde{\mathcal{C}}_{k,j-1}) = k, \\ \widetilde{b}_j^k + 1, & \text{if } m(\widetilde{\mathcal{C}}_{k,j-1}) > k, \end{cases} \quad \text{for } j \geq 2. \quad (3.60)$$

Since $m(\widetilde{\mathcal{C}}_{k,j}) \geq k$, we can use the same argument as for (3.21) to show that for each j there exist $h_j \leq h'_j$ such that $\widetilde{\mathcal{C}}_{k,j} = [\widehat{\omega}_{h_j}^{k-1}, \widehat{\alpha}_{h'_j}^{k-1}] \cap \Gamma$ (use (3.65) below instead of (3.11)). Again we avoid double indexing, but h_j, h'_j depend on the step.

We now define, for $j \geq 1$:

$$\widehat{\omega}_{j,0}^k = \begin{cases} \widehat{\omega}_{h_j-3}^{k-1}, & \text{if } m(\widetilde{\mathcal{C}}_{k,j}) = k, \\ \widehat{\omega}_{h_j}^{k-1}, & \text{if } m(\widetilde{\mathcal{C}}_{k,j}) > k, \end{cases} \quad (3.61)$$

and

$$\widehat{\alpha}_{j,0}^k = \begin{cases} \widehat{\alpha}_{\hat{s}(j,1)}^{k-1}, & \text{if } m(\widetilde{\mathcal{C}}_{k,j}) = k, \\ \widehat{\alpha}_{h'_j}^{k-1}, & \text{if } m(\widetilde{\mathcal{C}}_{k,j}) > k, \end{cases} \quad (3.62)$$

where $\hat{s}(j,i)$ is such that $\omega(\widetilde{\mathcal{C}}_{k,j}) + iL^k/3 \in \mathcal{H}_{\hat{s}(j,i)}^{k-1}$, for $1 \leq i \leq \widehat{b}_{j+1}^k - 1$, and $\omega(\widetilde{\mathcal{C}}_{k,0})$ is interpreted as zero. We also set, for $j \geq 1$, and if $m(\widetilde{\mathcal{C}}_{k,j}) = k$,

$$\widehat{\alpha}_{j,i}^k = \begin{cases} \widehat{\alpha}_{\hat{s}(j,i+1)}^{k-1}, & \text{if } 1 \leq i \leq \widehat{b}_{j+1}^k - 2, \\ \widehat{\omega}_{j+1,0}^k - 1, & \text{if } i = \widehat{b}_{j+1}^k - 1, \end{cases} \quad (3.63)$$

while if $m(\widetilde{\mathcal{C}}_{k,j}) > k$ or $j = 0$,

$$\widehat{\alpha}_{j,i}^k = \begin{cases} \widehat{\alpha}_{\hat{s}(j,i)}^{k-1}, & \text{if } 1 \leq i \leq \widehat{b}_{j+1}^k - 2, \\ \widehat{\omega}_{j+1,0}^k - 1, & \text{if } i = \widehat{b}_{j+1}^k - 1, \end{cases} \quad (3.64)$$

and finally, for $j \geq 0$,

$$\widehat{\omega}_{j,i}^k = \widehat{\alpha}_{j,i-1}^k + 1 \quad \text{if } 1 \leq i \leq \widehat{b}_{j+1}^k - 1. \quad (3.65)$$

We then set

$$\widehat{H}_{j,i}^k = \bigcup_{s: \widehat{\mathcal{H}}_s^{k-1} \subset [\widehat{\omega}_{j,i}^k, \widehat{\alpha}_{j,i}^k]} \widehat{H}_s^{k-1},$$

if $j \geq 0$, $0 \leq i \leq \widehat{b}_{j+1}^k - 1$. In the next lemma we relate the layers $\{\widehat{H}_{j,i}^k, j \geq 0, 0 \leq i \leq \widehat{b}_{j+1}^k - 1\}$ to the layers $\{\widetilde{H}_{j,i}^k, j \geq 0, 0 \leq i \leq \widehat{b}_{j+1}^k - 1\}$. In particular, we state that for fixed k the intervals $[\widehat{\alpha}_{j,i}^k, \widehat{\omega}_{j,i}^k]$, $j \geq 0, 0 \leq i \leq \widehat{b}_{j+1}^k - 1$ form a partition of \mathbb{Z}_+ . As before we can then re-label the layers $\{\widehat{H}_{j,i}^k, j \geq 0, 0 \leq i \leq \widehat{b}_{j+1}^k - 1\}$ as $\widehat{\mathbf{H}}^k = \{\widehat{H}_j^k, j \geq 1\}$ in increasing order, and define $\widehat{\omega}_j^k$ and $\widehat{\alpha}_j^k$ through the identity

$$\widehat{H}_j^k = \bigcup_{s: \widehat{H}_s^{k-1} \subset [\widehat{\omega}_j^k, \widehat{\alpha}_j^k]} \widehat{H}_s^{k-1},$$

for each j , and $\widehat{\mathcal{H}}_j^k = [\widehat{\omega}_j^k, \widehat{\alpha}_j^k]$. As we shall see in Lemma 3.3, many of the layers in \widehat{H}^k “nearly” coincide with a layer in H^k .

We shall need some properties of the reversed layers which parallel the properties in Lemma 3.1. First some definitions. The *support* of the reversed layer $\widehat{H}_{j,i}^t$ is the interval $[\widehat{\omega}_{j,i}^t, \widehat{\alpha}_{j,i}^t]$.

The reversed layer $\widehat{H}_{j,i}^t$ is called *good of type 2* if $1 \leq i \leq \widehat{b}_{j+1}^t - 1$, and *good of type 1* if $i = 0$ and $m(\widetilde{\mathcal{C}}_{t,j}) = t$. It is called *bad* if $i = 0$ and $m(\widetilde{\mathcal{C}}_{t,j}) > t$.

$$\begin{aligned}\widehat{A}_t &:= \sup\{(\widehat{\alpha}_j^t - \widehat{\omega}_j^t) : j \geq 1, \widehat{H}_j^t \text{ is a good reversed } t\text{-layer}\} \\ &= \sup\{(\widehat{\alpha}_{j,i}^t - \widehat{\omega}_{j,i}^t) : j \geq 0, 0 \leq i \leq \widehat{b}_{j+1}^t - 1, \widehat{H}_{j,i}^t \text{ is a good reversed } t\text{-layer}\}.\end{aligned}$$

Lemma 3.2. *Let $L \geq 108$ and let γ be an environment with $\chi(\gamma) = 0$. Then the following properties hold:*

for all $t, j \geq 1$ the support of a bad reversed $(t-1)$ -layer cannot intersect the interval

$$[1, \alpha(\widetilde{\mathcal{C}}_{t,1}) - 1] \text{ or any interval } [\omega(\widetilde{\mathcal{C}}_{t,j}) + 1, \alpha(\widetilde{\mathcal{C}}_{t,j+1}) - 1]; \quad (3.66)$$

$$\text{for all } t \geq 1, j \geq 0, 1 \leq i \leq \widehat{b}_{j+1}^t - 1, \widehat{H}_{s^t(j,i)}^{t-1} \text{ is a good reversed } (t-1)\text{-layer}; \quad (3.67)$$

$$\begin{aligned}\text{for } t \geq 1, j \geq 1, 1 \leq \ell \leq L/3 \text{ and } m(\widetilde{\mathcal{C}}_{t,j}) = t, \widehat{H}_{h_j - \ell}^{t-1} \\ \text{is a good reversed } (t-1)\text{-layer};\end{aligned} \quad (3.68)$$

$$\text{for } t \geq 1, \widehat{A}_t \leq 2L^t; \quad (3.69)$$

$$\widehat{\omega}_{j,i}^t \leq \widehat{\alpha}_{j,i}^t \text{ for } j \geq 0, 0 \leq i \leq \widehat{b}_{j+1}^t - 1; \quad (3.70)$$

$$\begin{aligned}\text{the intervals } [\widehat{\omega}_{j,i}^t, \widehat{\alpha}_{j,i}^t], j \geq 0, 0 \leq i \leq \widehat{b}_{j+1}^t - 1 \text{ (ordered as } (j,i) = (0,0), \dots, \\ (0, \widehat{b}_1^t - 1), (1,0), \dots, (1, \widehat{b}_2^t - 1), (2,0), \dots) \text{ form a partition of } \mathbb{Z}_+;\end{aligned} \quad (3.71)$$

$$\text{for } t \geq 1, 0 \leq i \leq \widehat{b}_{j+1}^t, \text{ each } [\widehat{\omega}_{j,i}^t, \widehat{\alpha}_{j,i}^t] \text{ is a union of intervals} \quad (3.72)$$

$$[\widehat{\omega}_{u,v}^{t-1}, \widehat{\alpha}_{u,v}^{t-1}] \text{ over a finite number of suitable pairs } (u,v)$$

$$\text{(for } t = 1 \text{ this condition is taken to be fulfilled by convention);} \quad (3.73)$$

$$\text{for each } j \geq 1, \widetilde{\mathcal{C}}_{t,j} \subset [\widehat{\omega}_{j,0}^t, \widehat{\alpha}_{j,0}^t]; \quad (3.74)$$

$$\text{for each } j \geq 1, \text{span}(\widetilde{\mathcal{C}}_{t,j}) = [\widehat{\omega}_{j,0}^t, \widehat{\alpha}_{j,0}^t] \text{ when } m(\widetilde{\mathcal{C}}_{t,j}) > t. \quad (3.75)$$

We shall not prove this lemma. Its proof is essentially the same as that of Lemma 3.1 with much tedious definition pushing. We also note without proof that the analogue of (3.44) for reversed layers is

$$h_{j+1} - h'_j \geq \frac{L}{6} \text{ for } j \geq 0.$$

We repeat that we always take $L \geq 108$ and γ such that $\chi(\gamma) = 0$. For $k \geq 1, j \geq 0$ and $0 \leq i \leq \widehat{b}_{j+1}^k - 1$ we define $r(k, j, i)$ to be the rank number of the layer $\widetilde{H}_{j,i}^k$ in the partition \mathcal{H}^k of \mathbb{Z}_+ , that is, $\widetilde{H}_{j,i}^k = \mathcal{H}_{r(k,j,i)}^k$. Similarly we define $\widehat{r}(k, j, i)$ to be the unique number such that $\widehat{H}_{j,i}^k = \widehat{\mathcal{H}}_{\widehat{r}(k,j,i)}^k$ for $k \geq 1, j \geq 0, 0 \leq i \leq \widehat{b}_{j+1}^k - 1$.

The next lemma shows the relation between the partitions \mathcal{H}^k and $\widehat{\mathcal{H}}^k$. Each interval in one of these partitions differs not too much from one of the intervals in the other partition.

This is especially important for the intervals which contain the clusters $\tilde{\mathcal{C}}_{k,j}$. If such a cluster has mass $m(\tilde{\mathcal{C}}_{k,j}) > k$, then there even exists in each of the partitions an interval equal to the span of $\tilde{\mathcal{C}}_{k,j}$ (see (3.11) and (3.75)).

Lemma 3.3. *For $k \geq 1, j \geq 0$ and $0 \leq i \leq b_{j+1}^k - 1$,*

$$r(k, j, i) = \sum_{\ell=0}^{j-1} b_{\ell+1}^k + i + 1 = \sum_{\ell=0}^{j-1} \tilde{b}_{\ell+1}^k + \sum_{\ell=0}^{j-1} I[m(\tilde{\mathcal{C}}_{k,\ell+1}) > k] + i + 1. \quad (3.76)$$

For $k \geq 1, j \geq 1$ and $0 \leq i \leq \hat{b}_{j+1}^k - 1$,

$$\hat{r}(k, j, i) = \sum_{\ell=0}^{j-1} \hat{b}_{\ell+1}^k + i + 1 = \sum_{\ell=0}^{j-1} \tilde{b}_{\ell+1}^k + 1 + \sum_{\ell=1}^{j-1} I[m(\tilde{\mathcal{C}}_{k,\ell}) > k] + i + 1 \quad (3.77)$$

If $j = 0$, then

$$r(k, 0, i) = i + 1 \text{ for } 0 \leq i \leq b_1^k - 1 \text{ and } \hat{r}(k, 0, i) = i + 1 \text{ for } 0 \leq i \leq \hat{b}_1^k - 1. \quad (3.78)$$

For all $k, u \geq 1$,

$$|\mathcal{H}_u^k \triangle \hat{\mathcal{H}}_u^k| \leq 20L^{k-1} \quad (3.79)$$

(\triangle denotes symmetric difference and $|\cdot|$ the cardinality).

Proof. Recall that \mathcal{H}^k is just the increasing rearrangement of the intervals $[\tilde{\alpha}_{j,i}^k, \tilde{\omega}_{j,i}^k]$ with $0 \leq i \leq b_{j+1}^k, j \geq 0$. The first equality in (3.77) is therefore immediate from (3.8) and the fact that there are exactly $b_{\ell+1}^k$ layers $\tilde{H}_{\ell,i}^k$ with $0 \leq i \leq b_{\ell+1}^k$. The second equality then follows from (3.17). The equalities in (3.77) follow in the same way from (3.71) and (3.60). The first equality in (3.78) is immediate, because \mathcal{H}^k begins with the supports of the layers $\tilde{H}_{0,i}^k, 0 \leq i \leq b_1^k - 1$. The second equality in (3.78) follows in a similar manner.

To prove (3.79) we note first that for $j \geq 1, 0 \leq i \leq (b_{j+1}^k \wedge \hat{b}_{j+1}^k) - 1$,

$$\hat{r}(k, j, i) = \sum_{\ell=0}^{j-1} \tilde{b}_{\ell+1}^k + 1 + \sum_{\ell=0}^{j-2} I[m(\tilde{\mathcal{C}}_{k,\ell+1}) > k] + i + 1 = r(k, j, i) + I[m(\tilde{\mathcal{C}}_{k,j}) = k], \quad (3.80)$$

by virtue of (3.77), (3.79). We further note that for all $k \geq 1, j \geq 0$ it holds

$$b_{j+1}^k \in \{\tilde{b}_{j+1}^k, \tilde{b}_{j+1}^k + 1\} \text{ as well as } \hat{b}_{j+1}^k \in \{\tilde{b}_{j+1}^k, \tilde{b}_{j+1}^k + 1\}. \quad (3.81)$$

Now fix $k \geq 1$. As (i, j) traverses all pairs $j \geq 0, 0 \leq i \leq b_{j+1}^k - 1$ in the order given in (3.8) $r(k, j, i)$ runs through the positive integers in order. Equivalently, $\tilde{\mathcal{H}}_{r(k,j,i)}^k$ runs through the intervals $\mathcal{H}_u^k, u = 1, 2, \dots$ in order. We now must distinguish different cases. Let $r(k, j, i) = u$ and hence $\tilde{\mathcal{H}}_u^k = \tilde{\mathcal{H}}_{j,i}^k$ for a certain (j, i) with $0 \leq i \leq b_{j+1}^k - 1$. Then we have the cases (i) $m(\tilde{\mathcal{C}}_{k,j}) = k, j \geq 1, 1 \leq i \leq b_{j+1}^k - 1$; (ii) $m(\tilde{\mathcal{C}}_{k,j}) = k, j \geq 1$ but $i = 0$; (iii) $m(\tilde{\mathcal{C}}_{k,j}) > k, j \geq 1, 0 \leq i \leq b_{j+1}^k - 1$; (iv) $j = 0$. To make the argument clear, let us start with case (iii), which seems to be the simplest case. In this case (3.80) shows that

$\widehat{r}(k, j, i) = r(k, j, i) = u$, but there is one proviso. We can apply (3.80) only if (j, i) is a legitimate pair, that is, if $i \leq \widehat{b}_{j+1}^k - 1$. However, by (3.55) and (3.60) $\widehat{b}_{j+1}^k = \widetilde{b}_{j+1}^k + 1$, and hence by (3.81) and $m(\widetilde{\mathcal{C}}_{k,j}) > k$, we automatically have $i \leq b_{j+1}^k - 1 \leq \widehat{b}_{j+1}^k - 1$ in case (iii) (recall that we have $i \leq b_{j+1}^k - 1$ by choice of i ; see (3.8)). Thus $\widehat{r}(k, j, i) = r(k, j, i)$ for all pairs (j, i) in case (iii). Consequently, (3.79) reduces in case (iii) to

$$|[\widetilde{\alpha}_{j,i}^k, \widetilde{\omega}_{j,i}^k] \triangle [\widehat{\omega}_{j,i}^k, \widehat{\alpha}_{j,i}^k]| \leq 20L^{k-1}. \quad (3.82)$$

We shall verify this inequality in some subcases below, but first let us consider cases (i) and (ii). In case (i) $\widehat{r}(k, j, i) = r(k, j, i) + 1 = u + 1$, so \widehat{H}_u^k is the predecessor of $\mathcal{H}_{j,i}^k$ in the order of (3.71), i.e. $\widehat{H}_u^k = \widehat{H}_{j,i-1}^k$. Thus in case (i) (3.79) reduces to

$$|[\widetilde{\alpha}_{j,i}^k, \widetilde{\omega}_{j,i}^k] \triangle [\widehat{\omega}_{j,i-1}^k, \widehat{\alpha}_{j,i-1}^k]| \leq 20L^{k-1}. \quad (3.83)$$

(Note that $(j, i-1)$ is a legitimate pair, because $i \leq b_{j+1}^k - 1$ together with (3.81) implies $i-1 \leq \widehat{b}_{j+1}^k - 1$.) Finally, in case (ii) with the further restriction $j \geq 1$ we have $\widehat{r}(k, j, 0) = r(k, j, 0) + 1$, and consequently we will have $\widehat{H}_{j',i'}^k = \widehat{H}_u^k$ for $u = r(k, j, 0)$ if (j', i') is the immediate predecessor of $(j, 0)$, that is, if $(j', i') = (j-1, \widehat{b}_j^k - 1)$. Thus case (ii) reduces to

$$|[\widetilde{\alpha}_{j,0}^k, \widetilde{\omega}_{j,0}^k] \triangle [\widehat{\omega}_{j-1, \widehat{b}_j^k - 1}^k, \widehat{\alpha}_{j-1, \widehat{b}_j^k - 1}^k]| \leq 20L^{k-1}. \quad (3.84)$$

Case (iv) with $j = 0$ needs special treatment, but this is easy by means of (3.78).

To complete the proof we now prove (3.79) in a few subcases. We shall only treat some typical examples. In all cases we shall use the estimate

$$|[a_1, b_1] \triangle [a_2, b_2]| \leq |a_1 - a_2| + |b_1 - b_2|, \quad (3.85)$$

which is valid for any intervals $[a_i, b_i]$, $i = 1, 2$. Thus to prove (3.82) and (3.79) in case (iii) it suffices to verify

$$|\widetilde{\alpha}_{j,i}^k - \widetilde{\omega}_{j,i}^k| + |\widetilde{\omega}_{j,i}^k - \widehat{\alpha}_{j,i}^k| \leq 20L^{k-1}. \quad (3.86)$$

Now consider the subcase of (iii)

$$k \geq 2, j \geq 1, m(\widetilde{\mathcal{C}}_{k,j}) > k, 2 \leq i \leq \widehat{b}_{j+1}^k - 2. \quad (3.87)$$

In this subcase

$$[\widetilde{\alpha}_{j,i}^k, \widetilde{\omega}_{j,i}^k] = [\omega_{s(j,i-1)}^{k-1} + 1, \omega_{s(j,i)}^{k-1}] \text{ and } [\widehat{\omega}_{j,i}^k, \widehat{\alpha}_{j,i}^k] = [\widehat{\alpha}_{\widehat{s}(j,i-1)}^{k-1} + 1, \widehat{\alpha}_{\widehat{s}(j,i)}^{k-1}]. \quad (3.88)$$

Moreover, by definition, $s(j, i)$ satisfies (3.23), while

$$\sup\{(\omega_j^{k-1} - \alpha_j^{k-1}) : j \geq 2, H_j^{k-1} \text{ is a good } (k-1)\text{-layer}\} \leq A^{k-1} \leq 2L^{k-1}$$

(by (3.42) and (3.32)). Therefore,

$$|\widetilde{\alpha}_{s(j,i)}^{k-1} - \omega(\widetilde{\mathcal{C}}_{k,j}) - (i-1)L^k/3| \leq 2L^{k-1} \text{ and } |\widetilde{\omega}_{s(j,i)}^{k-1} - \omega(\widetilde{\mathcal{C}}_{k,j}) - iL^k/3| \leq 2L^{k-1} \quad (3.89)$$

(see also (3.35)). In turn, this implies that the left hand side of (3.86) is changed by at most $4L^{k-1} + 1$ if we replace $\tilde{\alpha}_{j,i}^k$ and $\tilde{\omega}_{j,i}^k$ by $\omega(\tilde{\mathcal{C}}_{k,j}) + (i-1)L^k/3$ and $\omega(\tilde{\mathcal{C}}_{k,j}) + iL^k/3$, respectively. Similarly, in the subcase (3.87) it holds

$$\hat{\omega}_{\tilde{s}(j,i)}^{k-1} \leq \omega(\tilde{\mathcal{C}}_{k,j}) + iL^k/3 \leq \hat{\alpha}_{\tilde{s}(j,i)}^{k-1}, \quad (3.90)$$

so that we have

$$|\hat{\alpha}_{\tilde{s}(j,i)}^{k-1} - \omega(\tilde{\mathcal{C}}_{k,j}) - iL^k/3| \leq 2L^{k-1}. \quad (3.91)$$

Thus, if we replace $\hat{\omega}_{j,i}^k$ and $\hat{\alpha}_{j,i}^k$ by $\omega(\tilde{\mathcal{C}}_{k,j}) + (i-1)L^k/3$ and $\omega(\tilde{\mathcal{C}}_{k,j}) + iL^k/3$, respectively, then the left hand side of (3.86) is again not changed by more than $4L^{k-1} + 1$. These considerations prove (3.82), and hence (3.79), when (3.87) holds.

We also consider some subcases of case (ii). To verify that (3.84) holds in case (ii) we shall use the bounds

$$\omega(\tilde{\mathcal{C}}_{k,j}) = \omega_{i_j}^{k-1} \leq \omega_{i_j+3}^{k-1} \leq \omega_{i_j}^{k-1} + 3(2L^{k-1} + 1) \leq \omega(\tilde{\mathcal{C}}_{k,j}) + 3(2L^{k-1} + 1), \quad (3.92)$$

and

$$\hat{\omega}_{h_j}^{k-1} - 3(2L^{k-1} + 1) \leq \hat{\omega}_{h_j-3}^{k-1} \leq \hat{\omega}_{h_j}^{k-1} = \alpha(\tilde{\mathcal{C}}_{k,j}). \quad (3.93)$$

The first inequality in (3.92) is obvious and the second one follows by the argument for (3.39). The third inequality is again obvious since $\omega_{i_j} \leq \omega_{i'_j} = \omega(\tilde{\mathcal{C}}_{k,j})$. The inequalities in (3.93) follow by similar arguments.

Case (ii) is split into two subcases, namely (iia) $j \geq 2, m(\tilde{\mathcal{C}}_{k,j-1}) = k, m(\tilde{\mathcal{C}}_{k,j}) = k, i = 0$ and (iib) $j = 1$ or $(j \geq 2 \text{ and } m(\tilde{\mathcal{C}}_{k,j-1}) > k), m(\tilde{\mathcal{C}}_{k,j}) = k, i = 0$. In each subcase the further subcase $k = 1$ requires the use of the special definitions of Step 1. These will allow us to verify (3.79) without use of α^0 or ω^0 (i.e., a superscript of 0 combined with further subscripts) in intermediate steps. The steps for $k = 1$ are essentially the same as for $k \geq 2$ and we shall restrict ourselves here to the cases with $k \geq 2$.

In subcase (iia) we have $b_j^k = \tilde{b}_j^k = \hat{b}_j^k$ and $k \geq 2, j \geq 2$ and for suitable $\theta_\ell \in [-1, 1]$ (which may depend on k, j)

$$\begin{aligned} \tilde{\alpha}_{j,0}^k &= \tilde{\omega}_{j-1, b_j^k-1}^k + 1 = \omega_{s(j-1, b_j^k-1)}^{k-1} + 1 = \omega(\tilde{\mathcal{C}}_{k,j-1}) + (b_j^k - 1)L^k/3 + \theta_1(2L^{k-1} + 1), \\ \tilde{\omega}_{j,0}^k &= \omega_{i_j+3}^{k-1} = \omega(\tilde{\mathcal{C}}_{k,j}) + 3\theta_2(2L^{k-1} + 1) \text{ (by (3.92))}, \\ \hat{\omega}_{j-1, \hat{b}_j^k-1}^k &= \hat{\alpha}_{j-1, \hat{b}_j^k-2}^k + 1 = \hat{\alpha}_{\tilde{s}(j-1, \hat{b}_j^k-1)}^{k-1} + 1 = \omega(\tilde{\mathcal{C}}_{k,j-1}) + (\hat{b}_j^k - 1)L^k/3 + \theta_3(2L^{k-1} + 1), \\ \hat{\alpha}_{j-1, \hat{b}_j^k-1}^k &= \hat{\omega}_{j,0}^k - 1 = \hat{\omega}_{h_j-3}^{k-1} = \hat{\omega}_{h_j}^{k-1} + 3\theta_4(2L^{k-1} + 1) = \alpha(\tilde{\mathcal{C}}_{k,j}) + 3\theta_4(2L^{k-1} + 1). \end{aligned} \quad (3.94)$$

In subcase (iia) the inequality (3.84), and hence (3.79), follows from these relations together with $b_j^k = \hat{b}_j^k$ and $|\alpha(\tilde{\mathcal{C}}_{k,j}) - \omega(\tilde{\mathcal{C}}_{k,j})| \leq 3L^{k-1}$ (see (2.23)).

In subcase (iib) we have $b_j^k = \tilde{b}_j^k = \hat{b}_j^k - 1$. The first, second and fourth line of (3.95) need no change for this subcase, but in the third line we have to appeal to (3.64) instead of

(3.63). This third line now has to be replaced by

$$\widehat{\omega}_{j-1, \widehat{b}_j^k-1}^k = \widehat{\alpha}_{j-1, \widehat{b}_j^k-2}^k + 1 = \widehat{\alpha}_{\widehat{s}(j-1, \widehat{b}_j^k-2)}^{k-1} + 1 = \omega(\widetilde{\mathcal{C}}_{k,j-1}) + (\widehat{b}_j^k - 2)L^k/3 + \theta_3(2L^{k-1} + 1).$$

The inequalities (3.84) and (3.79) in case (iib) follow from these observations and $\widehat{b}_j^k - 2 = b_j^k - 1$. Any reader who has followed the proof so far will be able to complete the remaining cases. \square

The reversed sites $\widehat{S}_{u,v}^k$ are defined as in (3.46), with \mathcal{H}_j^k replaced by $\widehat{\mathcal{H}}_j^k$.

Remark 3.4. In view of the preceding lemma it becomes natural to write $\widehat{S}(S_{u,v}^k) := \widehat{S}_{u,v}^k$ as well as $S(\widehat{S}_{u,v}^k) := S_{u,v}^k$ for $(u, v) \in \widetilde{\mathbb{Z}}_+^2$.

4. PASSABLE SITES

At the end of this section we state key estimates that will lead to the proof of Theorem 1.1. Before doing that we introduce several key definitions: a rooted seed, passability from the seed (s -passable), and an open cluster.

s- and c-Passable sites. Rooted seed.

Step 0. A 0-site is called *s-passable* if and only if the site is open.

Rooted 0-seed. The rooted 0-seed $Q^{(0)}(u, v)$, with root at (u, v) , is the set of three open 0-sites in $\widetilde{\mathbb{Z}}_+^2$:

$$Q^{(0)} = Q^{(0)}(u, v) = \{(u, v), (u+1, v+1), (u-1, v+1)\}.$$

The site (u, v) is called the *root* of $Q_{u,v}^{(0)}$, and we write $R(Q^{(0)}) = \{(u, v)\}$; the sites $(u-1, v+1)$ and $(u+1, v+1)$ are called the *active sites* of $Q^{(0)}$, and we set $A(Q^{(0)}) = \{(u-1, v+1), (u+1, v+1)\}$. (When the location of the seed is not important we will suppress the subscript.)

We remind the reader that $\widetilde{\mathbb{Z}}_+^2$ is oriented upwards in the second coordinate. We shall therefore say that A is connected to B by an open path π only if π is an open path which respects the orientation and with initial and endpoint in A and B , respectively. We call such a path simply an *open path from A to B*. For $A \subset \widetilde{\mathbb{Z}}_+^2$ we call the *top line of A* the subset $\{(x, y) \in A : y = y_0\}$, where y_0 has the maximal value for which this subset is nonempty. If y_0 takes the smallest value for which $\{(x, y) \in A : y = y_0\}$ is non empty, then we call this subset the *bottom line* of A . Note that the top line and bottom line as defined here are not complete lines, not even intervals, in general.

We next define open clusters, passability and rooted seeds for a general $k \geq 1$. These definitions have to be used in sequence. First we must use them to define rooted 0-seed and open cluster of a 0-site; then passability of a good 1-site (this relies on the definition of a rooted 0-seed and its open cluster already given above); then a rooted 1-seed and the open cluster of a 1-site; next passability of a good 2-site and a rooted 2-seed, etc.

Let $\theta(p)$ denote the percolation probability for the homogeneous Bernoulli oriented percolation model with parameter p :

$$\theta(p) := P_p\{\text{there exists an infinite open path from the origin}\}. \quad (4.1)$$

We first take p_G large enough so that $\theta(p_G) > 1/2$, and ρ will be some fixed number in the interval $(1/2, \theta(p_G))$. The constant c was chosen in (3.45).

Open cluster of a rooted 0-seed. The open cluster of a rooted 0-seed $Q^{(0)} = Q_{u,v}^{(0)}$ is the collection of 0-sites w for which there exists an open path of 0-sites from $A(Q^{(0)})$ to w . This open cluster is denoted by $U(Q^{(0)})$.

We shall soon need the open cluster of a 0-seed $Q^{(0)}$ restricted to the kernel of a 1-site S^1 which is located such that all 0-sites of $A(Q^{(0)})$ are below and adjacent to the middle third of S^1 , i.e., adjacent to $F(S^1)$. This will simply be the collection of good 0-sites w for which there exists an open path of good 0-sites from $A(Q^{(0)})$ to w and inside S^1 . Note that only the last restriction is added to the definition of the open cluster of $Q^{(0)}$.

s-Passable k -site. A good k -site S^k is said to be *s-passable from a rooted $(k-1)$ -seed $Q^{(k-1)}$* if the following conditions (i), (iis) and (iiis) are satisfied (see Figure 5)

- (s1) All 0-sites of $A(Q^{(k-1)})$ are below and adjacent to the middle third of the bottom layer of S^k , i.e., adjacent to $F(S^k)$ (see (3.54) for the definition of $F(S^k)$).
- (s2) There exist two rooted $(k-1)$ -seeds, $\tilde{Q}_l^{(k-1)}$ and $\tilde{Q}_r^{(k-1)}$ say, such that their top lines are contained in the top line of $D_l(S^k)$ and the top line of $D_r(S^k)$, respectively, and such that there exist open oriented paths of 0-sites, entirely contained in S^k , from 0-sites adjacent to $A(Q^{(k-1)})$ to $R(\tilde{Q}_l^{(k-1)})$ and to $R(\tilde{Q}_r^{(k-1)})$.
- (s3) The open cluster of $Q^{(k-1)}$ restricted to $\text{Ker}(S^k)$ contains at least $\rho cL/12$ $(k-1)$ -sites in each of $D_l^K(S^k)$ and $D_r^K(S^k)$ (see (3.48) for the definition of D_θ^K).

Remark. We shall denote the leftmost rooted $(k-1)$ -seed which fulfills the requirements for $\tilde{Q}_l^{(k-1)}$ in (s2) as $Q_l(S^k)$. Similarly $Q_r(S^k)$ denotes the rightmost rooted $(k-1)$ -seed which fulfills the requirements for $\tilde{Q}_r^{(k-1)}$. We further define $A(S^k) = A(Q_l(S^k)) \cup A(Q_r(S^k))$ and call the sites in this set the *active sites of S^k* . Note that in these definitions $Q_l(S^k)$, $Q_r(S^k)$ and $A(S^k)$ also depend on $Q^{(k-1)}$, even though the notation does not indicate this. However, in the definition below of the open cluster of a rooted k seed we shall use the more explicit notation $Q_\theta(S^k, Q^{(k-1)})$ with $\theta = l$ or r to indicate this dependence.

Rooted k -seed. A rooted k -seed is formed by a rooted $(k-1)$ -seed $Q^{(k-1)}$ and three good k -sites

$$S_{u,v}^k, S_{u-1,v+1}^k \text{ and } S_{u+1,v+1}^k;$$

such that

- (i) $S_{u,v}^k$ is s-passable from $Q^{(k-1)}$,
- (ii) $S_{u-1,v+1}^k$ and $S_{u+1,v+1}^k$ are passable from $Q_l(S_{u,v}^k)$ and $Q_r(S_{u,v}^k)$, respectively.

The corresponding k -seed is denoted by

$$Q^{(k)} = S_{u,v}^k \cup S_{u-1,v+1}^k \cup S_{u+1,v+1}^k \cup Q^{(k-1)}.$$

We set

$$\begin{aligned} R(Q^{(k)}) &= R(Q^{(k-1)}), \\ A(Q^{(k)}) &= A(Q_l(S_{u-1,v+1}^k)) \cup A(Q_r(S_{u-1,v+1}^k)) \\ &\quad \cup A(Q_l(S_{u+1,v+1}^k)) \cup A(Q_r(S_{u+1,v+1}^k)). \end{aligned}$$

The 0-site $R(Q^{(k)})$ is called the root of $Q^{(k)}$; the sites in $A(Q^{(k)})$ are called the active sites of $Q^{(k)}$.

Remark. We point out that the locations of $D_l(S_{u,v}^k)$ and $D_r(S_{u,v}^k)$ are such that the definition of a rooted k -seed makes sense. Specifically, the top line of $D_l(S_{u,v}^k)$ is adjacent to and just below $F(S_{u-1,v+1}^k)$ and so, if $S_{u,v}^k$ is s -passable, then also $Q_l(S_{u,v}^k)$ is adjacent to and just below $F(S_{u-1,v+1}^k)$. Thus, it makes sense to speak of s -passability of $S_{u-1,v+1}^k$ from $Q_l(S_{u,v}^k)$. Similar statements hold for $S_{u+1,v+1}^k$ and $Q_r(S_{u,v}^k)$.

Remark. Note that in the definition of a rooted 0-seed we required the three 0-sites which make up the seed to be open. Starting from this fact we deduce the following lemma.

Lemma 4.1. *In a rooted k -seed $Q^{(k)}$ there exists for each $x \in A(Q^{(k)})$ an open oriented path of 0-sites from $R(Q^{(k)})$ to x .*

Proof. We use a proof by induction on k . For $k = 0$ the conclusion of the lemma is obvious. For the induction step, let $k \geq 1$ and assume that the conclusion of the lemma with k replaced by $k - 1$ has already been proven. Let further $Q^{(k)} = S_{u,v}^k \cup S_{u-1,v+1}^k \cup S_{u+1,v+1}^k \cup Q^{(k-1)}$ be a rooted k -seed and let $x \in A(Q_l(S_{u-1,v+1}^k))$. The other possible locations for x in $A(Q^{(k)})$ can be handled in the same way. Then there exist open paths of 0-sites π_i as follows:

- π_1 from $y := R(Q_l(S_{u-1,v+1}^k))$ to x (by the induction hypothesis);
- π_2 from some point z in $A(Q_l(S_{u,v}^k))$ to y (because $S_{u-1,v+1}^k$ is s -passable from $Q_l(S_{u,v}^k)$);
- π_3 from $w := R(Q_l(S_{u,v}^k))$ to z (by the induction hypothesis again);
- π_4 from some point a in $A(Q^{(k-1)})$ to w (because $S_{u,v}^k$ is s -passable from $Q^{(k-1)}$);
- π_5 from $R(Q^{(k-1)})$ to a (by the induction hypothesis once more).

Now concatenation of the paths $\pi_5, \pi_4, \dots, \pi_1$ gives an open path of 0-sites from $R(Q^{(k-1)})$ to x , as desired. \square

Remark. If the origin is connected to $R(Q^{(k-1)})$ by an open path, and S^k is s -passable from $Q^{(k-1)}$, it follows that the origin is connected by an open path of 0-sites to all sites in $A(S^k)$.

Open cluster of a rooted k -seed with $k \geq 1$. The open cluster of a rooted k -seed $Q^{(k)} = S_{u,v}^k \cup S_{u-1,v+1}^k \cup S_{u+1,v+1}^k \cup Q^{(k-1)}$ is defined as the collection of k -sites consisting of $S_{u,v}^k, S_{u-1,v+1}^k, S_{u+1,v+1}^k$ and the k -sites S for which there exists a sequence $S(1), \dots, S(n)$ of k -sites with the following properties:

$$\text{each } S(j) \text{ is good,} \tag{4.2}$$

$$S(n) = S, \tag{4.3}$$

for $0 \leq j \leq n$, $S(j)$ is s -passable from a rooted $(k-1)$ -seed $\tilde{Q}(j-1)$, (4.4)

where, in the notation of the remark following condition (iiis), (4.5)

$$\tilde{Q}(j-1) = Q_{\theta(j-1)}^{(k-1)}(S(j-1), Q_{\theta(j-2)}^{(k-1)}(S(j-2))). \quad (4.6)$$

Here the $\theta(i)$ can be l or r , independently of each other, and $S(-1)$ is interpreted as $S_{u,v}^k$, and $S(0) = S_{u+\phi, v+1}^k$ with $\phi = -1$ if $\theta(0) = l$ and $\phi = +1$ if $\theta(0) = r$. Also, $\tilde{Q}(-1) = Q^{(k-1)}$. We define the *open cluster restricted to $\text{Ker}(S^{k+1})$ of the rooted k -seed $Q^{(k)}$* in the same way as the open cluster of $Q^{(k)}$, but now with the added restriction that $Q^{(k)}$ and all $S(j)$, $-1 \leq j \leq n$, are contained in $\text{Ker}(S^{k+1}) \cap \dot{S}^{k+1}$ (see definition (3.51) for Ker and (3.47) for k).

c -Passable k -site. A good 0-site is said to be *c-passable* if it is open.

For $k \geq 1$, a good k -site S^k is said to be *c-passable* if:

- (c1) There exist two rooted $(k-1)$ -seeds, $\tilde{Q}_l^{(k-1)}$ and $\tilde{Q}_r^{(k-1)}$ say, such that their top lines are contained in the top line of $D_l(S^k)$ and the top line of $D_r(S^k)$, respectively, and such that there exist open oriented paths of 0-sites, entirely contained in S^k , from the lowest 0-level layer of $F(S^k)$ to $R(\tilde{Q}_l^{(k-1)})$ and to $R(\tilde{Q}_r^{(k-1)})$.
- (c2) The open cluster of the lowest 0-level layer of $F(S^k)$ restricted to $\text{Ker}(S^k)$ contains at least $\rho cL/12$ $(k-1)$ -sites in each of $D_l^{\mathcal{K}}(S^k)$ and $D_r^{\mathcal{K}}(S^k)$ (see (3.48) for the definition of $D_{\theta}^{\mathcal{K}}$).

Definition 4.2. We say that a good k -site S^k has an *s-dense kernel* from a seed $Q^{(k-1)}$ if condition (s3) holds. If (c2) holds we say that S^k has *c-dense kernel*.

Remark. Taking into account the reversed partition, we analogously define the notions of \hat{c} - and \hat{s} -passable sites.

Lemma 4.3. (a) Let $[\alpha_v^k, \omega_v^k]$ be a k -interval and let \mathcal{K}_v^k be defined as in (3.49) and (3.50). If \tilde{I}^{k-1} is a $(k-1)$ -interval contained in \mathcal{K}_v^k , then \tilde{I}^{k-1} is also good.

(b) Let $[\alpha_v^k, \omega_v^k]$ be good of type 1 and let $\tilde{\mathcal{C}}_{k,j}$ be the unique cluster of \mathbf{C}_k of mass at least k in $[\tilde{\alpha}_{j,0}^k, \tilde{\omega}_{j,0}^k]$. If $[\tilde{\alpha}_{q,i}^{k-1}, \tilde{\omega}_{q,i}^{k-1}]$ is the last $(k-1)$ -interval below $[\tilde{\alpha}_{j,0}^k, \tilde{\omega}_{j,0}^k]$, then this $(k-1)$ -interval is good of type 2.

Remark. Roughly speaking part (a) says that a $(k-1)$ -layer in the kernel of a good k -layer is again good.

Proof. (a) We give a proof by contradiction. So assume that \tilde{I}^{k-1} is a bad $(k-1)$ -interval. Then it contains a cluster $\tilde{\mathcal{C}}_{k-1,j} \in \mathbf{C}_{k-1}$ for some j , of mass at least k and level at most $k-1$. Either $\tilde{\mathcal{C}}_{k-1,j} \in \mathbf{C}_k$, or $\tilde{\mathcal{C}}_{k-1,j}$ is a constituent of some cluster in \mathbf{C}_k . In any case, \tilde{I}^{k-1} intersects a cluster $\tilde{\mathcal{C}}_{k,p} \in \mathbf{C}_k$ of mass $\geq k$, for some p . Moreover, our choice of the partition \mathbf{H}^k is such that for $i \geq 1$ the interval $[\tilde{\alpha}_{p,i}^k, \tilde{\omega}_{p,i}^k]$ does not intersect any cluster of mass $\geq k$, while for $i = 0$, $[\tilde{\alpha}_{p,0}^k, \tilde{\omega}_{p,0}^k]$ contains $\tilde{\mathcal{C}}_{k,p}$ and no other cluster of mass $\geq k$ (see (3.10) and

(3.8)). However, we assumed that $\tilde{I}^{k-1} \subset [\alpha_v^k, \omega_v^k]$, so that $\tilde{\mathcal{C}}_{k,p} \cap [\alpha_v^k, \omega_v^k]$ is non-empty. This implies that $[\alpha_v^k, \omega_v^k]$ equals $[\tilde{\alpha}_{p,i}^k, \tilde{\omega}_{p,i}^k]$ for some $i \geq 0$.

As we already stated, for $i \geq 1$, $[\tilde{\alpha}_{p,i}^k, \tilde{\omega}_{p,i}^k]$ does not intersect any cluster of mass $\geq k$, so that $i \geq 1$ is incompatible with a non-empty intersection of $\tilde{I}^{k-1} \subset [\tilde{\alpha}_{p,i}^k, \tilde{\omega}_{p,i}^k]$ with $\tilde{\mathcal{C}}_{k,p}$. But also the case $i = 0$ is impossible by (3.49) and (3.50). Indeed, if (3.49) applies, then $\tilde{I}^{k-1} \subset [\alpha_v^k, \omega_v^k]$ cannot intersect any cluster of mass $\geq k$, by virtue of (3.31) (note that the hypothesis $\chi(\gamma) = 0$ is not needed for (3.31)). On the other hand, if (3.50) applies we must have that $\tilde{I}^{k-1} \subset \mathcal{K}_v^k$ lies strictly below $\tilde{\mathcal{C}}_{k,p}$ and therefore does not intersect $\tilde{\mathcal{C}}_{k,p}$. (b) Since $[\tilde{\alpha}_{j,0}^k, \tilde{\omega}_{j,0}^k]$ is good of type 1 it contains exactly one cluster \mathbf{C}_k of mass at least k , and this cluster has mass k (see the remark following (3.43)). In our previous notation this cluster is denoted as $\tilde{\mathcal{C}}_{k,j}$. By (3.11) $\text{span}(\tilde{\mathcal{C}}_{k,j}) = [\tilde{\alpha}_{p,0}^{k-1}, \tilde{\omega}_{p,0}^{k-1}]$ for some p . The $(k-1)$ -interval preceding this is $[\tilde{\alpha}_{p-1,b_p^{k-1}-1}^{k-1}, \tilde{\omega}_{p-1,b_p^{k-1}-1}^{k-1}]$ (see (3.8)). We then must have $(q, i) = (p-1, b_p^{k-1}-1)$. Since $b_p^{k-1} \geq \tilde{b}_p^{k-1} \geq 3$, $[\tilde{\alpha}_{p-1,b_p^{k-1}-1}^{k-1}, \tilde{\omega}_{p-1,b_p^{k-1}-1}^{k-1}]$ is a good $(k-1)$ -interval of type 2 (see the lines following (3.29)). \square

Lemma 4.4. *Let $k \geq 1$. If a good k -site S^k has an s -dense kernel from a rooted seed $Q^{(k-1)}$, then $A(Q^{(k-1)})$ is connected by open paths of 0-sites inside $\text{Ker}(S^k)$ to at least $(\rho c L / 12)^k$ 0-sites in the top line of $\text{Ker}(S^k)$, but to the left (respectively to the right) of the middle third of this top line.*

Proof. The proof goes by induction on k . Start with $k = 1$. If S^1 has an s -dense kernel from the 0-rooted seed $Q^{(0)}$, then there are at least $\rho c L / 12$ 0-sites in the open cluster of $Q^{(0)}$ restricted to $\text{Ker}(S^1)$ and in $D_\theta^K, \theta = l, r$. Each such 0-site is just a vertex w for which there is an path in $\text{Ker}(S^1)$ of open good 0-sites from $A(Q^{(0)})$ to w and in $D_\theta^K, \theta = l$ or r . Moreover, such w automatically lie in the top line of $\text{Ker}(S_{u,v}^1)$, because for $k = 1$ the cardinality of $\mathcal{D}_{j,i}^{1,K}$ equals 1 for each (j, i) with $m(\tilde{\mathcal{C}}_{1,j}) = 1, 0 \leq i \leq b_{j+1}^1 - 1$ (see (3.13), (3.14) and (3.52)). Thus, for $k = 1$, the conclusion of the lemma is immediate from the definitions of an s -dense kernel and of the open cluster of $Q^{(0)}$.

Now assume that the lemma has already been proven for k replaced by $k-1$. Assume further that S^k has an s -dense kernel from the rooted seed $Q^{(k-1)} = S_{u,v}^{k-1} \cup S_{u-1,v+1}^{k-1} \cup S_{u+1,v+1}^{k-1} \cup Q^{(k-2)}$. Let \tilde{S}^{k-1} be a $(k-1)$ -site which belongs to the open cluster of $Q^{(k-1)}$. Further, for the sake of argument, let \tilde{S}^{k-1} lie in $D_l^K(S^k)$. Then there exists some n and sequences $S^{k-1}(0), \dots, S^{k-1}(n)$ and $\theta(0), \dots, \theta(n)$ such that (4.2)-(4.6) with k replaced by $k-1$ and S by \tilde{S}^{k-1} hold. In particular, $S^{k-1}(j)$ is passable from the rooted $(k-2)$ -seed $\tilde{Q}(j-1) = Q_{\theta(j-1)}^{(k-2)}(S^{k-1}(j-1), Q_{\theta(j-2)}^{(k-2)}(S^{k-1}(j-2)))$. Also, part of the definition of s -passability gives that the top line of $\tilde{Q}(j-1)$ will be equal to the top line of $S^{k-1}(j-1)$. A simple induction argument (with respect to j), similar to the proof of Lemma 4.1, then shows that there exists a path of open 0-sites in $\text{Ker}(S^k)$ from $A(Q^{(k-1)})$ to $R(\tilde{Q}(j))$. For $j = n$, an application of Lemma 4.1 then shows that for each vertex x in $A(\tilde{Q}(n))$ there

exists an open path of 0-sites from $A(Q^{(k-1)})$ to x . Since S^k has a dense kernel there are at least $\rho cL/12$ choices for \tilde{S}^{k-1} which are contained in $D_l^K(S^k)$ (respectively in $D_r^K(S^k)$). Since different $(k-1)$ -sites are disjoint, there are at least $\rho cL/12$ disjoint choices for \tilde{S}^{k-1} in each of $D_l^K(S^k)$ and $D_r^K(S^k)$. Moreover, if $\tilde{S}^{k-1} = S^{k-1}(n)$ is any fixed one of the possible choices, then \tilde{S}^{k-1} is s -passable from a rooted seed $\tilde{Q}(n-1)$, as we just showed. By the induction hypothesis, there exist at least $[\rho cL/12]^{k-1}$ 0-sites y in the top line of $\text{Ker}(\tilde{S}^{k-1})$, with the property that there exists an open path in $\text{Ker}(\tilde{S}^{k-1})$ from some $x \in A(\tilde{Q}(n-1))$ to y . Such a connection can be concatenated with the connection from $A(Q^{(k-1)})$ to x , to obtain an open path in $\text{Ker}(S^k)$ from $A(Q^{(k-1)})$ to y . But then there are at least $[\rho cL/12]^{k-1}$ choices for y in each possible \tilde{S}^{k-1} and $\rho cL/12$ choices for \tilde{S}^{k-1} . In total this gives at least $[\rho cL/12]^k$ 0-sites with the required open connection from $R(A^{(k-1)})$.

The 0-sites y constructed in the preceding paragraph lie in the top line of $\text{Ker}(\tilde{S}^{k-1})$ for some \tilde{S}^{k-1} , which itself lies in $D_l^K(S^k) \cup D_r^K(S^k)$. It remains to show that these y lie in the top line of $\text{Ker}(S^k)$ itself. However, Lemma 4.3 shows that

$$\text{each of the possible } \tilde{S}^{k-1} \text{ is a good } (k-1)\text{-site of type 2.} \quad (4.7)$$

Now, as observed right after the definition (3.51), $\text{Ker}(\tilde{S}^{k-1})$ equals \tilde{S}^{k-1} if (4.7) holds. Thus (4.7) implies that the possible y lie in the top lines of the possible \tilde{S}^{k-1} and, as we shall show now, these latter top lines are contained in the top line of $\text{Ker}(S^k)$. This is so because the projection on the vertical axis of \tilde{S}^{k-1} is a whole interval of the partition \mathcal{H}^{k-1} of \mathbb{Z}_+ , and the same is true for the projection on the vertical axis of D_θ^K , $\theta = l$ or r (see (3.46), (3.13), (3.14), (3.25), (3.28)). Since $\tilde{S}^{k-1} \subset D_\theta^K$, the projections of \tilde{S}^{k-1} and D_θ^K must be equal (in fact the projections of D_l^K and D_r^K are trivially equal; see (3.48)). Thus the top line of \tilde{S}^{k-1} must equal the top line of $D_\theta^K(S^k)$, and this equals the top line of $\text{Ker}(S^k)$ by (3.52). This completes the proof of the induction step and the lemma. \square

Definition 4.5. For $\mathcal{C} \subset \mathbb{Z}_+$ we write

$$B(\mathcal{C}) = \{(x, y) \in \tilde{\mathbb{Z}}_+^2 : y \in \text{span}(\mathcal{C})\}.$$

We use $\mathcal{C}(m, \ell)$ to denote a generic cluster of level ℓ , i.e. an element of $\mathbf{C}_{\ell, \ell}$, with mass m . The corresponding horizontal layer $B(\mathcal{C}(m, \ell))$ is called the *bad layer of mass m and level ℓ associated with $\mathcal{C}(m, \ell)$* . If there is no ambiguity to which cluster $\mathcal{C}(m, \ell)$ of mass m and level ℓ we are associating the bad layer we will use $B(m, \ell)$ instead of $B(\mathcal{C}(m, \ell))$.

Definition 4.6. (*Matching pair*) Let $B(m, \ell)$ be as above. For any k , $\ell - 1 \leq k \leq m - 1$, we say that two good k -sites $S_{(u, v-1)}^k, \hat{S}_{(u', v'+1)}^k$ form a matching pair with respect to $B(m, \ell)$ if either $u' = u$ or $u' = u \pm 1$, according as $v' - v$ is even or odd.

At this point we are ready to give a more detailed description of the inductive step. The environment will be a fixed γ with $\chi(\gamma) = 0$. We assume $p_G > 2/3$ and N is a fixed integer

for which $[3(1 - p_G)]^{N/5-2} \leq 1/72$. This implies that

$$8(1 - p^3)^{N/5} \leq (1 - p)^2, \text{ for all } p \geq p_G. \quad (4.8)$$

If $k \geq 1$ and S^k is a good k -site of type 1 we will be looking at a very particular way to obtain its s -passability from a given $(k - 1)$ -seed $Q^{(k-1)}$. It will turn out to be enough for Theorem 1.1 to consider the situation when S^k is good of type 1 and the bad $(k - 1)$ -sites contained in S^k lie in a layer $B(k, \ell)$, for some $\ell < k$. $\text{Span}(\mathcal{C}(k, \ell))$, the projection on the vertical axis of $B(k, \ell)$, equals $[\alpha_v^{k-1}, \omega_v^{k-1}]$ for some v in this case. The kernel of S^k will be the part of S^k which lies strictly below the horizontal line $y = \alpha_v^{k-1}$ and the top line of the kernel is contained in the line $y = \alpha_v^{k-1} - 1$. The $(k - 1)$ -sites with their top line equal to the top line of the kernel are the $(k - 1)$ -sites with projection onto the vertical axis equal to $\mathcal{H}_{v-1}^{k-1} = [\alpha_{v-1}^{k-1}, \omega_{v-1}^{k-1}]$. These are therefore of the form $S_{u,v-1}^{k-1}$ for some u .

In each case, passability of S^k will be built from the occurrence of three events W_1^k, W_2^k and W_3^k which we define now. Further properties of these W_i^k will be given in Theorem 4.11 at the end of this section.

For $k \geq 1$

$$W_1^k(s) = W_1^k(s, S^k, Q_0^{(k-1)}) = \{S^k \text{ has } s\text{-dense kernel from a seed } Q_0^{(k-1)}\},$$

where $Q_0^{(k-1)}$ is a given rooted $(k - 1)$ -seed which fulfills condition (i) for s -passability of S^k . If $W_1^k(s)$ occurs, then there exists for $\theta = l$ (left) and for $\theta = r$ (right) in D_θ^K a collection \mathcal{R}_θ^{k-1} of at least $\theta cL/12$ $(k - 1)$ -sites S^{k-1} in the open cluster of $Q_0^{(k-1)}$. Each of these is s -passable from some rooted $(k - 2)$ -seed. We remind the reader that this implies that each of the S^{k-1} in \mathcal{R}_θ^{k-1} contains for $\lambda = l$ and for $\lambda = r$ a rooted $(k - 2)$ -seed $Q_\lambda^{(k-2)} = Q_\lambda^{k-2}(S^{k-1})$ with top line contained in $D_\lambda^K(S^{k-1})$ for which there exists an open path of 0-sites in $\text{Ker}(S^k)$ from $A(Q_0^{(k-1)})$ to $R(Q_\lambda^{(k-2)})$ (see the proof of Lemma 4.3). The union of the active sites of $Q_l^{(k-2)}(S^{k-1})$ and $Q_r^{(k-2)}(S^{k-1})$ is denoted by $A(S^{k-1})$.

For $k \geq 2$ the event $W_2^k(s)$ occurs if and only if $W_1^k(s)$ occurs and for $\theta = l$ and for $\theta = r$ there exist a collection \mathcal{L}_θ^{k-1} of $(k - 1)$ -sites with the following properties:

- (i) $\mathcal{L}_\theta^{k-1} \subset \mathcal{R}_\theta^{k-1}$ and the cardinality of \mathcal{L}_θ is at least N ;
- (ii) for each $S^{k-1} = S_{u,v-1}^{k-1} \in \mathcal{L}_\theta$ there exists an index p with $|p - u| \leq 2$ and a rooted $(k - 2)$ -seed $Q^{(k-2)}(p)$ say, in $S_{p,v+1}^{k-1}$ and with top line contained in the $D_l(S_{p,v+1}^{k-1}) \cup D_r(S_{p,v+1}^{k-1})$ and such that there is an open path inside S^k from $A(S^{k-1})$ to $R(Q^{(k-2)}(p))$.

When $k = 1$ we modify (ii) somewhat because $Q^{(k-2)}$ is meaningless in this case. Recall that a 0-site is just a vertex of $\tilde{\mathbb{Z}}_+^2$. For $k = 1$ \mathcal{R}_θ^0 will just be taken as the collection of 0-sites $(u, v - 1)$ in D_θ^K for which there exists an open path in $\text{Ker}(S^k)$ from $A(Q_0^{k-1})$ to $(u, v - 1)$. We then replace (ii) by

(ii, k=1) for each $S_{u,v-1}^0 = (u, v - 1) \in \mathcal{L}_\theta$, there exists a p with $|p - u| \leq 2$ such that there is an open path in S^1 from $(u, v - 1)$ to $S_{p,v+1}^0 = (p, v + 1)$.

Finally, if $W_2^k(s)$ occurs, and $k \geq 2$, then $W_3^k(s)$ occurs if and only if there exist (at least) two rooted $(k-1)$ -seeds in S^k , $Q_{l,1}^{k-1}$ with top line in $D_l(S^k)$ and $Q_{r,1}^{k-1}$ with top line in $D_r(S^k)$, and open connections of 0-sites in S^k from the collection of the rooted $(k-2)$ -seeds $Q^{(k-2)}(p)$ mentioned in (ii) above to $R(Q_{l,1}^{k-1})$ as well as to $R(Q_{r,1}^{k-1})$. When $k=1$ we merely replace the collection of rooted $(k-2)$ -seeds $Q^{(k-2)}(p)$ here by the collection of 0-sites $S_{p,v+1}^0$ mentioned in (ii, $k=1$).

The definitions of the W_i^k are unfortunately very involved. The reader should think of W_1^k as providing open connections from the middle third of the bottom of S^k to the top of its kernel; then W_2^k will provide open connections from the top of the kernel to the top of the bad layer, and finally W_3^k from the top of the bad layer to the top of S^k . The connections required for W_2^k from the bottom of the bad layer to its top are the most difficult to come by. They will be constructed in the next section.

Before formulating our basic set of estimates we state a number of properties of supercritical oriented site percolation on $\tilde{\mathbb{Z}}_+^2$. We start with a simple observation which holds for any Bernoulli percolation as an immediate consequence of coupling.

Lemma 4.7. *Consider site percolation on a graph \mathcal{G} (possibly partially oriented). Denote the probability measure under which all sites are independently open with probability p by P_p , and let \mathcal{E} be some increasing event. If $p_0, p'_0 \in [0, 1]$ and $\tilde{p} = 1 - (1 - p_0)(1 - p'_0)$, then*

$$P_p\{\mathcal{E}\} \geq 1 - (1 - P_{p_0}\{\mathcal{E}\})(1 - P_{p'_0}\{\mathcal{E}\}) \quad \text{for all } p \geq \tilde{p}. \quad (4.9)$$

Now let us go back to oriented site percolation on $\tilde{\mathbb{Z}}_+^2$ and let P_p be as in the preceding lemma. For $\mathcal{A} \subset \tilde{\mathbb{Z}}_+^2$ define

$$\Theta(\mathcal{A}) = \{\text{all vertices in } \mathcal{A} \text{ are open}\} \quad (4.10)$$

and let $|\mathcal{A}|$ denotes the cardinality of \mathcal{A} .

Lemma 4.8. *There exists some $\tilde{p} \in (0, 1)$ and a universal constant $c_5 < \infty$ such that for all $p \geq \tilde{p}$ and all subsets \mathcal{A} of $2\mathbb{Z} \times \{0\}$ it holds*

$$P_p\{\text{there is an open path from } \mathcal{A} \text{ to } \infty \mid \Theta(\mathcal{A})\} \geq 1 - c_5[9(1-p)]^{|\mathcal{A}|+1}. \quad (4.11)$$

Proof. The lemma states that the conditional probability, given $\Theta(\mathcal{A})$, that percolation occurs from at least one site in $|\mathcal{A}|$ is at least $1 - c_5(1-p)^{|\mathcal{A}|+1}$. We shall only need this if \mathcal{A} is an interval of a integers, and therefore we shall prove (4.11) only in this case. However [6] (p. 1029) proves that this is the worst case, i.e., that if (4.11) holds for \mathcal{A} an interval, then it holds in general. ([6] discusses bond percolation, but a small modification of his argument works for site percolation.)

Now let $\mathcal{A} = \{0, 2, \dots, 2(a-1)\} \times \{0\}$ and let \mathcal{F} be the collection of sites $(x, y) \in \tilde{\mathbb{Z}}_+^2$ for which there exists an open path from \mathcal{A} to (x, y) (with (x, y) itself also open). Then

$$1 - P_p\{\text{there is an open path from } \mathcal{A} \text{ to } \infty \mid \text{all of } \mathcal{A} \text{ is open}\} \leq P_p\left\{\bigcup_F \{\mathcal{F} = F\}\right\}, \quad (4.12)$$

where the union runs over all finite connected subsets F of $\tilde{\mathbb{Z}}_+^2$ which contain all of \mathcal{A} . We bound the right hand side of (4.12) by the usual contour method, as we explain now. As in Section 10 of [6] or [14], let D be the diamond $\{(x, y) : |x| + |y| \leq 1\} \subset \mathbb{R}^2$. For F a finite connected subset of $\tilde{\mathbb{Z}}_+^2$ which contains \mathcal{A} , we define $\tilde{F} = F + D$ and $\Gamma(F)$ = the topological boundary of the infinite component of $\mathbb{R}^2 \setminus \tilde{F}$. Then $\Gamma(F)$ is made up of edges of the lattice $\mathbb{Z}_{\text{odd}}^2 := \{(x, y) \in \mathbb{Z}^2 : x + y \text{ is odd}\}$ and it separates $\mathcal{A} \subset F$ from infinity. Suppose that $\mathcal{F} = F$ occurs and that e is an edge between two vertices of $\{(x, y) \in \mathbb{Z}^2 : x + y \text{ is even}\}$ which crosses one of the sides of one of the diamonds $v + D, v \in F$. In fact we must then have that one endpoint of e equals v and the other endpoint, w say, lies in the unbounded component of $\mathbb{R}^2 \setminus \tilde{F}$. There are then two possibilities. Either

$$w \text{ lies below } v, \quad (4.13)$$

so that a path on $\tilde{\mathbb{Z}}_+^2$ is prevented from going from v to w by the orientation of $\tilde{\mathbb{Z}}_+^2$. Or,

$$w \text{ lies above } v, \quad (4.14)$$

in which case w must be closed (otherwise w would belong to F , since an open path to v can be continued by going along e from v to w). It follows from this argument that the event $\cup_F \{\mathcal{F} = F\}$ is contained in the event that there exists some contour Γ made up of sides of the diamonds $u + D, u \in \tilde{\mathbb{Z}}_+^2$, which separates \mathcal{A} from infinity, and which has the following property: if the edge $\{v, w\}$ crosses one of the sides which make up Γ and $v \in \text{interior}(\Gamma)$ and $w \in \text{exterior}(\Gamma) \cap \tilde{\mathbb{Z}}_+^2$ and (4.14) holds, then w is closed. Consequently, the right hand side of (4.12) is bounded by

$$\sum_{\Gamma} P_p \{\text{each } w \in \tilde{\mathbb{Z}}_+^2 \text{ as above for which (4.14) holds is vacant}\}. \quad (4.15)$$

It is shown in [14] and [6] that the number of w for which (4.14) holds is at least $|\Gamma|/2$, where $|\Gamma|$ denotes the number of edges in Γ . Moreover, as one traverses the line $\{x = y + 2i\}$, starting at $(2i, 0) \in \mathcal{A}$ and increasing x (and y), the first vertex $w \in \tilde{\mathbb{Z}}_+^2$ in the unbounded component of $\mathbb{R}^2 \setminus \Gamma$ which one meets has to be closed. Since this holds for every $0 \leq i \leq a-1$, the number of w for which (4.14) holds is at least a . In fact, there have to be at least $a+1$ such vertices w , because the first vertex w on the line $x = -y$ which lies in the unbounded component of $\mathbb{R}^2 \setminus F$ also satisfies (4.14), but does not lie on any of the lines $x = y + 2i$. It follows that the term in (4.15) for a specific Γ is at most $(1-p)^{(|\Gamma|/2) \vee (a+1)}$. Moreover, the number of possible Γ with $|\Gamma| = n$ is at most 3^{n-1} , because each possible Γ which separates \mathcal{A} from infinity must contain the lower left edge of the diamond $(0, 0) + D$, centered at the origin. It follows that (4.15), and hence also the right hand side of (4.12) is bounded by

$$\sum_{n=1}^{\infty} 3^{n-1} (1-p)^{(n/2) \vee (a+1)}.$$

The lemma follows. □

Again consider oriented site percolation on $\tilde{\mathbb{Z}}_+^2$. Write $\mathbf{0}$ for the origin and define

$$\begin{aligned} r_n &:= \sup\{(x, n) \in \tilde{\mathbb{Z}}_+^2 : \text{there exists an open path from } \mathbf{0} \text{ to } (x, n)\}, \\ \ell_n &:= -\inf\{(x, n) \in \tilde{\mathbb{Z}}_+^2 : \text{there exists an open path from } \mathbf{0} \text{ to } (x, n)\}, \end{aligned} \quad (4.16)$$

$$r_n = \ell_n = 0 \text{ if there is no open path from } \mathbf{0} \text{ to } \mathbb{Z} \times n.$$

We further remind the reader that the percolation probability $\theta(p)$ was defined in (4.1). It is known (see [6]) that for all $p \geq \tilde{p} > p_c$ we have $\theta(p) \geq \theta(\tilde{p}) > 0$ and there exists $s(p) \in (0, +\infty)$ for which

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} r_n &= \lim_{n \rightarrow \infty} -\frac{1}{n} \ell_n = s(p) \text{ a.s. } [P_p] \text{ on the event} \\ \Omega_0 &:= \{\text{there exists an open path from } \mathbf{0} \text{ to } \infty\}; \end{aligned} \quad (4.17)$$

$s(p)$ is called the edge-speed (see [6] or [14]).

Finally we need the existence of a positive density in $[-ns(p), ns(p)] \times \{n\}$ of sites which have an open connection from a fixed finite nonempty set. The next lemma gives the precise meaning of this statement. We need the following definition: Let $\alpha \leq \beta$ and $\eta > 0$. Also let $\mathcal{A} = \{0, 2, 4, \dots, 2a - 2\}$ be some nonempty interval of a even integers. Then

$$\begin{aligned} \nu_n(\alpha, \beta) &= \nu_n(\alpha, \beta, \mathcal{A}, \eta) := \text{number of points } (x, n) \text{ with } \alpha n \leq x \leq \beta n, \\ &\quad x + n \text{ even, for which there exists an open path from} \\ &\quad \mathcal{A} \times \{0\} \text{ to } (x, n) \text{ which stays inside } [-\eta n, \eta n] \times [0, n]. \end{aligned} \quad (4.18)$$

Lemma 4.9. *Let $0 < \varepsilon, \eta \leq 1$. There exists some $\bar{p} = \bar{p}(\varepsilon, \eta) < 1$ such that for $p \geq \bar{p}$ there exists an $n_0 = n_0(\varepsilon, \eta, p)$, such that for $n \geq n_0$ and $-s(p) \leq \alpha \leq \beta \leq s(p)$,*

$$P_p \left\{ \frac{1}{n} \nu_n(\alpha, \beta, \mathcal{A}, \eta) \geq [\theta(p)(\beta - \alpha) - \varepsilon] \frac{\eta}{17} \text{ for all } -s(p) \leq \alpha \leq \beta \leq s(p) \right\} \geq 1 - \varepsilon. \quad (4.19)$$

Proof. In general, and in particular in the definitions (4.17) and (4.19) of ν_n , an open path has to have its initial point and its endpoint open. For the sake of the proof of the present lemma we shall call a path open if all its vertices other than its initial point are open. Until the last three sentences of the proof we allow its initial point to be open or closed.

Clearly $\nu_n(\alpha, \beta, \mathcal{A}, \eta)$ is increasing in \mathcal{A} , so that it suffices to prove (4.19) for $\mathcal{A} =$ the origin. We shall restrict ourselves to $p \geq \tilde{p}$ as in Lemma 4.8. By obvious monotonicity we then have $\theta(p) \geq \theta(\tilde{p}) > 0$. In addition it is immediate from the definition (4.18) that $s(p) \leq 1$. In fact

$$r_n \leq n \text{ and } \ell_n \leq n \text{ for all } n. \quad (4.20)$$

Thus, it holds

$$\theta(p) \geq \theta(\tilde{p}) > 0 \text{ and } 0 < s(\tilde{p}) \leq s(p) \leq 1 \quad (4.21)$$

for the p which we are considering.

Now let $\varepsilon > 0$ and $\eta > 0$ be given. We define

$$m = m(n, \eta) = \left\lfloor \frac{\eta}{8} n \right\rfloor, k_0 = k_0(\eta) = \left\lfloor \frac{n}{m} \right\rfloor - 1, m' = n - k_0 m. \quad (4.22)$$

Finally, we choose ε_1 such that

$$0 < \varepsilon_1 \leq \frac{\varepsilon}{2k_0}. \quad (4.23)$$

and then $\bar{p} = \bar{p}(\varepsilon, \eta) < 1$ so that $\bar{p} \geq \tilde{p} \vee (1 - \varepsilon/2)$ and

$$\theta(\bar{p}) := P_{\bar{p}}\{\Omega_0\} \geq 1 - \varepsilon_1. \quad (4.24)$$

Such a $\bar{p} < 1$ exists by (4.11).

First we observe that (4.18) implies that for every $p \geq \bar{p}$ and $\eta_1 > 0$ there exists a constant $c_6 = c_6(\varepsilon_1, \eta_1, p)$ such that

$$\begin{aligned} & P_p\{|r_t - ts(p)| > c_6 + \eta_1 t \text{ or } |\ell_t + ts(p)| > c_6 + \eta_1 t \text{ for some } t \in \mathbb{Z}_+\} \\ & \leq P_p\{|r_t - ts(p)| > c_6 + \eta_1 t \text{ or } |\ell_t + ts(p)| > c_6 + \eta_1 t \text{ for some } t \in \mathbb{Z}_+\} \cap \Omega_0 + P_p\{\Omega_0^c\} \\ & \leq 2\varepsilon_1. \end{aligned} \quad (4.25)$$

We observe next that if Ω_0 occurs, then r_m and ℓ_m are well defined for all t . Furthermore, for any m there must exist open paths $\pi_\ell = \pi_\ell(m)$ and $\pi_r = \pi_r(m)$ from the origin to (ℓ_m, m) and to (r_m, m) , respectively, and these paths must lie in $[-m, m] \times [0, m]$ (see (4.20)). Next let $x \in \mathbb{Z}$ with $x + m$ even be such that $-\ell_m \leq x \leq r_m$. Consider the open paths starting at (x, m) going *downwards*, that is against the orientation on $\tilde{\mathbb{Z}}_+^2$ assumed so far. Assume that for a given $x \in [-\ell_m, r_m]$ there exists an infinite downward open path, $\tilde{\pi}_x$ say, starting at (x, m) . Since this path starts between $(-\ell_m, m)$ and (r_m, m) , it must hit $\pi_\ell \cup \pi_r$. Furthermore, the path $\tilde{\pi}_x$ necessarily stays in $[x - m, x + m] \times [0, m]$ up till time m . For the sake of argument, let $\tilde{\pi}_x$ first intersect π_ℓ in a point (y, q) with $0 \leq q \leq m$. Then the piece of π_ℓ from the origin to (y, q) , followed by the piece of $\tilde{\pi}_x$, traversed in the forward direction, from (y, q) to (x, m) forms an open oriented path from the origin to (x, m) . A similar argument applies if $\tilde{\pi}_x$ hits π_r . Thus, if there exists a downward infinite open path from (x, m) , then there exists an open path from the origin to (x, m) . By the estimates on the locations of π_ℓ, π_r and of $\tilde{\pi}_x$ which we have just given, this path must be contained in $[-2m, 2m] \times [0, m]$.

Let us write J_x for the indicator function of the event that there is an open path contained in $[-2m, 2m] \times [0, m]$ from the origin to (x, m) . Also, let \tilde{I}_x and I_x be the indicator functions of the events that there exists an infinite open backwards path from (x, m) , respectively an infinite open forwards path from $(x, 0)$, which stays in $[-2m, 2m]$ during $[0, m]$. The preceding argument shows that for any $M \geq 0$, on the event

$$\mathcal{E}(M, m) := \cap\{-\ell_m \leq -2M \leq 2M \leq r_m\}, \quad (4.26)$$

it holds

$$\begin{aligned} \tilde{\nu}_m(M, \mathbf{0}) &:= \text{number of points } (x, m) \text{ with } x \in [-2M, 2M], \\ & x + m \text{ even, for which there exists an open path from } \mathbf{0} \\ & \text{to } (x, m) \text{ which stays inside } [-2m, 2m] \times [0, m] \\ & \geq \sum_{\substack{-2M \leq x \leq 2M \\ x+m \text{ even}}} J_x \geq \sum_{\substack{-2M \leq x \leq 2M \\ x+m \text{ even}}} \tilde{I}_x. \end{aligned} \quad (4.27)$$

Now, the monotonicity of $s(\cdot)$ and (4.21) and (4.18) imply that for each fixed $M > 0$ there exists an $m_1 = m_1(\varepsilon_1, M)$ such that for $m \geq m_1$ and $p \geq \bar{p}$

$$P_p\{\mathcal{E}(M, m) \text{ fails}\} \leq P_p\{\mathcal{E}(M, m) \text{ fails}\} \leq \varepsilon_1. \quad (4.28)$$

Further, the joint distribution of the \tilde{I}_x , $x + m$ even is the same as the joint distribution of the I_x , x even. Also, if

$$M + c_6 \leq \frac{m}{2} \text{ and } |x| \leq \frac{m}{2}, \quad (4.29)$$

then

$$I_x \geq K_x := I[\text{there exists an open path from } (x, 0) \text{ to infinity which} \\ \text{stays in } [x - M - c_6 - t, x + M + c_6 + t] \text{ for all } t \geq 0].$$

By the ergodic theorem (see (4.1) for θ)

$$\liminf_{M \rightarrow \infty} \frac{1}{M} \sum_{\substack{x \in [-2M, 2M], \\ x \text{ even}}} I_x \geq \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{\substack{x \in [-2M, 2M], \\ x \text{ even}}} K_x \geq 2\theta(p) \text{ a.s. } [P_p]. \quad (4.30)$$

Thus there exists an $M_0 = M_0(\varepsilon_1)$ such that for all $p \in [\bar{p}, 1)$

$$P_p\left\{\tilde{I}_x = 1 \text{ for some } x \in [-2M_0, 2M_0]\right\} \quad (4.31)$$

$$\geq P_{\bar{p}}\left\{\frac{1}{M_0} \sum_{\substack{x \in [-2M_0, 2M_0] \\ x+m \text{ even}}} \tilde{I}_x \geq (2 - \varepsilon_1)\theta\right\} \geq 1 - \varepsilon_1. \quad (4.32)$$

We take $m_2 = m_2(\varepsilon_1)$ such that $m_2/2 \geq 2M_0 + c_6$. Then (4.29) with M_0 for M holds true for any $|x| \leq 2M_0, m \geq m_2$.

We now apply (4.28), (4.28) and (4.32) to obtain for all $m \geq m_2, p \geq p_3$

$$P_p\{\text{there is at least one } x \in [-2M_0, 2M_0] \text{ with an open path from} \\ \mathbf{0} \text{ to } (x, m) \text{ which is contained in } [-2m, 2m] \times [0, m]\} \\ \geq 1 - 2\varepsilon_1.$$

In other words, if we first determine the state of all vertices (x, y) with $0 \leq y \leq m$, we will find with probability $1 - 2\varepsilon_1$ at least one vertex $(x_1, m) \in [-2M_0, 2M_0]$ with $x_1 + m$ even and with an open connection from $\mathbf{0}$ to (x_1, m) which stays in $[-2m, 2m] \times [0, m]$. On the event that such an x exists, let x_1 be the smallest x in $[-2M_0, 2M_0]$ with these properties. We can then repeat the argument (after a shift by (x_1, m)), to find that with a further conditional probability of at least $1 - 2\varepsilon_1$, there exists an $x_2 \in [x_1 - 2M_0, x_1 + 2M_0] \subset [-4M_0, 4M_0]$ with an open path from x_1 to x_2 which stays in $[x_1 - 2m, x_1 + 2m] \subset [-4m, 4m]$ during $[m, 2m]$. Concatenation of the open path from $\mathbf{0}$ to x_1 and the path from x_1 to x_2 gives an open path from $\mathbf{0}$ to $(x_2, 2m)$ which stays in $[-4m, 4m]$ during $[0, 2m]$. Similarly, we find by repeating the argument k_0 times that there is a probability of at least $(1 - 2\varepsilon_1)^{k_0}$ that $\mathbf{0}$ is connected by an open path which stays in $[-2k_0m, 2k_0m] \times [0, k_0m]$ to a vertex (x_{k_0}, k_0m) with $|x_{k_0}| \leq 2k_0M_0$.

We need to concatenate paths once more. This time we replace m by $m' \in [m, 2m]$ (see (4.22)) and the sum over $-2M_0 \leq x \leq 2M_0$ in (4.30) by the sum over $\alpha m' \leq x \leq \beta m'$ for some fixed $-s(p) \leq \alpha \leq \beta \leq s(p)$. In essentially the same way as before we conclude that for $p \geq \bar{p}$ and $m \geq m_3 = m_3(\varepsilon_1, p)$ (suitable)

$$\begin{aligned} P_p \{ & \text{for all } -s(p) \leq \alpha \leq \beta \leq s(p) \text{ there are at least } [\theta(p)(\beta - \alpha)/2 - \varepsilon_1]m' \\ & \text{values of } x \text{ with } \alpha m' \leq x \leq \beta m', x + m' \text{ even, for which there} \\ & \text{is an open path from } \mathbf{0} \text{ to } (x, m') \text{ which stays inside} \\ & [-2m', 2m'] \times [0, m'] \text{ during } [0, m'] \} \\ & \geq 1 - 2\varepsilon_1. \end{aligned}$$

If x_{k_0} as described above exists, then there is a conditional probability, given the state of all vertices $(x, y) \in \tilde{\mathbb{Z}}_+^2$ with $y \leq k_0 m$, of at least $(1 - 2\varepsilon_1)$ that $(x_{k_0}, k_0 m)$ is connected to at least

$$[\theta(\beta - \alpha)/2 - \varepsilon_1]m' \geq [\theta(\beta - \alpha)/2 - \varepsilon_1]m \geq [\theta(\beta - \alpha)/2 - \varepsilon_1] \left\lfloor \frac{\eta}{8} n \right\rfloor$$

vertices $(x', km + m') = (x', n)$ in $[\alpha m' - 2k_0 m, \beta m' + 2k_0 m] \times n$ by open paths which stay in

$$[-2k_0 M_0 - 2m', 2k_0 M_0 + 2m'] \times [0, n] \text{ during } [0, k_0 m + m'] = [0, n].$$

But by (4.22) there exists some $n_0 = n_0(\varepsilon, \eta)$ such that for $n \geq n_0$ it holds $m \geq m_1 \vee m_2 \vee m_3$ and

$$2k_0 M_0 + 2m' \leq 2k_0 M_0 + 4m \leq 2 \frac{n}{m} M_0 + 4 \frac{\eta}{8} n \leq \eta n,$$

so that the constructed paths stay in $[-\eta n, \eta n] \times [0, n]$, as is required for them to be counted in ν_n . Also, by our choice of ε_1 in (4.23)

$$(1 - 2\varepsilon_1)^{k_0+1} \geq 1 - 2(k_0 + 1)\varepsilon_1 \geq 1 - \varepsilon/2.$$

We had to concatenate $k_0 + 1$ paths, each of which existed with a conditional probability of at least $1 - 2\varepsilon_1$, given the previously chosen paths. Thus the whole construction works with a probability of at least $(1 - \varepsilon_1)^{k_0+1} \geq 1 - \varepsilon/2$. This proves (4.19) when $\mathcal{A} = \mathbf{0}$. As pointed out before this proves the lemma if we do not insist that the starting point of an open path is open. However, if we revert to our previous convention that an open path must have an open initial and final point, then our construction of open paths from $\mathbf{0}$ to the horizontal line $\{y = n\}$ is valid only on the event $\{\mathbf{0} \text{ is open}\}$. We therefore have to discard the event $\{\mathbf{0} \text{ is closed}\}$. Correspondingly, the probability of finding the required open paths is at least $(1 - \varepsilon_1)^{k_0+1} - (1 - p) \geq 1 - \varepsilon/2 - \varepsilon/2 = 1 - \varepsilon$ (recall that $p \geq \bar{p} \geq 1 - \varepsilon/2$; see the line before (4.24)). \square

Corollary 4.10. *Let $\Theta(\mathcal{A})$ be as in (4.10) and*

$$\Omega(\mathcal{A}) := \{\text{there is an open path from } \mathcal{A} \text{ to infinity}\}.$$

Then under the conditions of Lemma 4.9, if $n \geq n_0$, it holds for any finite set $\mathcal{A} \subset \tilde{\mathbb{Z}}_+^2$

$$P_p \left\{ \frac{1}{n} \nu_n(\alpha, \beta, \mathcal{A}, \eta) \geq [\theta(p)(\beta - \alpha) - \varepsilon] \frac{\eta}{17} \text{ for all } -s(p) \leq \alpha \leq \beta \leq s(p) \mid \Theta(\mathcal{A}) \right\} \geq 1 - \varepsilon. \quad (4.33)$$

and

$$P_p \left\{ \frac{1}{n} \nu_n(\alpha, \beta, \mathcal{A}, \eta) \geq [\theta(p)(\beta - \alpha) - \varepsilon] \frac{\eta}{17} \text{ for all } -s(p) \leq \alpha \leq \beta \leq s(p) \mid \Omega(\mathcal{A}) \right\} \geq 1 - \varepsilon. \quad (4.34)$$

Proof. Since $\Theta(\mathcal{A})$ and $\Omega(\mathcal{A})$ are increasing events of the environment, these inequalities are immediate from (4.19) and the Harris-FKG inequality. \square

The W_i^k in the next theorem were defined in the paragraph following (4.8). In this theorem the environment is fixed at γ and the estimates are uniform in γ . The probability in this theorem refers only to the choice of the occupation variables, once the nature (good or bad) of each site has been fixed. $\chi(\gamma)$ is defined in (2.34).

Theorem 4.11. *Let $p_B > 0$. Then there exist some $p^* = p^*(p_B) < 1$ and $L_1 = L_1(p_B, p_G) < \infty$ such that for $p_G \geq p^*$ and $L \geq L_1$ and for every γ with $\chi(\gamma) = 0$, every $k \geq 1$ and good k -site of type 1 which intersects exactly one bad layer $B(k, \ell)$ (for some $\ell \leq k-1$), the following bounds hold:*

(a) *If $Q^{(k-1)}$ is a rooted $(k-1)$ -seed and S^k a good of type 1 k -site which satisfy condition (i) for an s -passable k -site, then*

$$P\{W_1^k(s) \mid Q^{(k-1)} \text{ is a rooted } (k-1)\text{-seed}\} \geq 1 - \frac{(1 - p_G)^{k+1}}{4}. \quad (4.35)$$

(b)

$$P\{W_2^k \mid W_1^k\} \geq 1 - \frac{(1 - p_G)^{k+1}}{4}. \quad (4.36)$$

(c)

$$P\{W_3^k \mid W_2^k\} \geq 1 - \frac{(1 - p_G)^{k+1}}{4}. \quad (4.37)$$

(d)

$$\begin{aligned} P\{S^k \text{ is } s\text{-passable from } Q^{(k-1)} \mid Q^{(k-1)} \text{ is a rooted } (k-1)\text{-seed}\} &\geq 1 - (1 - p_G)^{k+1}, \\ P\{S^k \text{ is } c\text{-passable}\} &\geq 1 - (1 - p_G)^{k+1}. \end{aligned} \quad (4.38)$$

Proof. The proof goes by induction. For simplicity we write $(a_k), (b_k), (c_k)$ and (d_k) for the corresponding statement at level k . After proving $(a_1), (b_1)$ and (c_1) we shall prove here the following implications:

$$(a_k), (b_k) \text{ and } (c_k) \text{ together} \implies (d_k); \quad (4.39)$$

$$(d_k) \implies (a_{k+1}) \text{ and } (c_{k+1}). \quad (4.40)$$

To complete the proof we shall show in the next section that (a_j) for $j \leq k+1$ and $(b_j), (c_j)$ for $j \leq k$ imply (b_{k+1}) .

We first observe that (4.39) is immediate from the definitions.

Now start with $k = 1$. For (a_1) , assume for the sake of argument that $S^1 = S_{0,v}^1 = ((-cL/2, cL/2] \times \mathcal{H}_v^1) \cap \tilde{\mathbb{Z}}_+^2$ (see (3.46)) with kernel $S_{0,v}^1 \cap (\mathbb{Z} \times \mathcal{K}_v^1)$ (see (3.51)). \mathcal{K}^1 is some interval, $[y_0, y_1]$ say.

Since S^1 is assumed to be good of type 1, it contains a unique cluster of mass ≥ 1 and this cluster has to be a single bad line. This cluster will be $\mathcal{C}_{1,j}$ for some j in the notation of Section 2. Since $\text{Ker}(S_1)$ is the part of S^1 below the bad line which intersects S^1 , this bad line is the line $y = y_1 + 1$. The bad lines before that will be $\mathcal{C}_{1,j-1}$ (or there will not be a previous bad line if $j = 1$). In this situation $y_1 - y_0$ is $\alpha(\mathcal{C}_{1,j}) - 1 - \omega(\mathcal{C}_{1,j-1}) \geq L - 4$ (see (2.3) and (3.2); if $j = 1$ we have to use the assumption $\chi(\gamma) = 0$ instead of (2.3). In any case, $y_1 - y_0 \geq L/4$ if L is taken ≥ 6 , and we can take L as large as desired if we take $\delta > 0$ small enough (see Lemma 2.1). In the other direction, S^1 is a good site of level 1, so $y_1 - y_0 \leq 2L$ by virtue of (3.34).

We also point out that all horizontal lines $\mathbb{Z} \times \{y\}$ with $y_0 \leq y \leq y_1$ lie in $\text{Ker}(S^1)$ and are good lines.

The condition that $Q^{(0)}$ is a rooted 0-seed gives us two adjacent open vertices $(x, y_0 - 1)$ and $(x + 2, y_0 - 1)$ for some odd $x \in [-cL/6 - 1, cL/6 + 1]$. For $W^1(s, S^1, Q^{(0)})$ to hold it suffices to have open paths in $\text{Ker}(S^1)$ from $(x, y_0 - 1) \cup (x + 2, y_0 - 1)$ to at least $\rho cL/12$ 0-sites in $D_\theta^\mathcal{K}$ for $\theta = l$ and $\theta = r$. In the simple case of $k = 1$ these are just open paths in S^1 to $[-\frac{5}{12}cL, -\frac{1}{3}cL] \times \{y_1\}$ (if $\theta = l$) and to $[\frac{1}{3}cL, \frac{5}{12}cL] \times \{y_1\}$. (4.35) for $k = 1$ can now be satisfied for large L by an application of Lemma 4.9 that guarantees positive density of the open oriented cluster restricted to S^1 at a suitable height proportional to L , and then using unrestricted growth to achieve density ρ in $D_\theta^\mathcal{K}$ for $\theta = l$ and $\theta = r$.

Next, (4.36) for $k = 1$ is easy. If $W_1^1(s)$ occurs, then for $\theta = l, r$ there exist sets \mathcal{R}_θ^0 containing at least $\rho cL/(12)$ open 0-sites which have an open connection from the origin. These sets are contained in the top line of $\text{Ker}(S^1)$, that is, in the horizontal line $\{y = y_1 - 1\}$. It is important that these sets \mathcal{R}_θ^0 are determined by the occupation variables $\eta_{(a,b)}$ with $b \leq y_1 - 1$. For $W_2^1(s)$ to occur, there should be at least N (see (4.8)) sites (a, y_1) in each of $\mathcal{R}_\theta^0, \theta = l$ or r which have an open connection to $(x, y_1 + 2)$ (which is the line just above the bad line $\{y = y_1 + 1\}$) for some $x \in [a - 2, a + 2]$. But the cardinality of \mathcal{R}_θ^0 is at least $\rho cL/12$ and hence goes to infinity with L . Thus if we keep $0 < p_B, p_G$ and N fixed, then the conditional probability in the left hand side of (4.36) tends to 1 as $L \rightarrow \infty$. Indeed, given $W_1^0(s)$, the event that (a, y_1) has an open connection to $[a - 2, a + 2] \times \{y_1 + 2\}$ has a strictly positive conditional probability, and these events for $a = a'$ and $a = a''$ are conditionally independent when $|a' - a''| \geq 5$. Thus, by raising L_1 if necessary, (4.36) follows.

We turn to (4.37). If $W_2^1(s)$ occurs, then let \mathcal{L}_θ^0 be the subset of 0-sites (a, y_1) in \mathcal{R}_θ^0 which have an open connection to $(x, y_1 + 2)$ for some $x \in [a - 2, a + 2]$. On the event $W_2^1(s)$ the cardinality of \mathcal{L}_θ^0 is at least N . Denote the collection of 0-sites $(x, y_1 + 2)$ with an open connection from some $(a, y_1) \in \mathcal{L}_\theta^0$ as just mentioned $\tilde{\mathcal{L}}_\theta^0$. On the event $W_2^1(s)$, the

cardinality of $\tilde{\mathcal{L}}_\theta^0$ is at least $N/5$. Then $W_3^1(s)$ occurs if for $\theta = l$ as well as $\theta = r$, there is a $(x, y_1 + 2) \in \tilde{\mathcal{L}}_\theta^0$ which has an open connection to a rooted 0-seed with top line in the top line of S^1 . Note that the top line of S^1 is contained in the line $\{y = y_1 + 4\}$, by virtue of (3.5). Therefore, if $(x, y_1 + 2) \in \tilde{\mathcal{L}}_\theta^0$, then the conditional probability that it has such an open connection is bounded below by p_G^3 . Since the cardinality of $\tilde{\mathcal{L}}_\theta^0$ is at least $N/5$ one easily sees that $P\{W_2^1(s) \mid W_1^1(s)\} \rightarrow 1$ as $N \rightarrow \infty$. In fact, (4.8) suffices to guarantee (4.37). Thus if we first pick N so that (4.8) holds and then L_1 so that (4.36) holds then both (4.36) and (4.37) hold.

If $p_G > 2/3$ and N has been chosen as above, we see at once from (4.8) that (d_k) implies (c_{k+1}) . A coupling argument (Lemma 4.7) easily shows that (a_{k+1}) follows from (d_k) . It remains to show that having $(a_j), j \leq k+1$ and $(b_j), (c_j), j \leq k$ we get (b_{k+1}) . This is the core of the proof and we postpone it to Sect. 6. It requires a more detailed study of clusters introduced in Sect.2, and which is the object of the next section.

5. TOWARDS DRILLING. STRUCTURE OF CLUSTERS

Lemma 5.1. *If \mathcal{C} is a cluster of mass m and level ℓ , then it has at most $m - \ell + 1$ constituents.*

Proof. Assume that \mathcal{C} is formed from an ℓ -run of r constituents: $\mathcal{C}_1, \dots, \mathcal{C}_r$, $r \geq 2$, each \mathcal{C}_i being of level $\ell_i < \ell$, and mass m_i . From the definition of ℓ -run we see that $m_i \geq \ell$, $i = 1, \dots, r$. On the other hand using the definition of the mass of a cluster (first equality in (2.11)) we see that $m \geq m_1 + (r - 1)$. The statement follows at once. \square

Notation. Given $\Gamma(\omega) = \{x \in \mathbb{Z}_+ : \xi_x = 1\}$ and an interval $[a, b]$ we denote by $\Gamma_{[a,b]}$ a new configuration on \mathbb{Z}_+ : $\Gamma_{[a,b]} \equiv \Gamma_{[a,b]}(\omega) = \Gamma(\omega) \cap [a, b]$. Equivalently, $\xi_{[a,b]}(x) = \xi_x$ if $x \in [a, b]$, and is zero otherwise.

Definition 5.2. (Porous medium) *We say that the segment $[x_1, x_2]$ is porous medium of level k (with respect to Γ) if:*

- 1) $\mathbf{C}_\infty(\Gamma_{[x_1, x_2]})$ contains no clusters of mass strictly larger than k ;
- 2) for any $\mathcal{C} \in \mathbf{C}_\infty(\Gamma_{[x_1, x_2]})$ we have:

$$d(\mathcal{C}, x_1) \geq L^{m(\mathcal{C})} - 1 \quad \text{and} \quad d(\mathcal{C}, x_2) \geq L^{m(\mathcal{C})} - 1.$$

In particular, $x_1, x_2 \notin \Gamma$. When $k = 0$ the definition reduces to $\Gamma \cap [x_1, x_2] = \emptyset$.

Lemma 5.3. *a) If $\ell \geq 1$ and $\mathcal{C}_i, \mathcal{C}_j \in \mathbf{C}_{\ell-1}(\Gamma)$ are two consecutive constituents of an ℓ -run, then the interval $[\omega(\mathcal{C}_i) + 1, \alpha(\mathcal{C}_j) - 1]$ is a porous medium of level $(\ell - 1)$ with respect to Γ .
b) If $k \geq 1$ and $\mathcal{C}_i, \mathcal{C}_j \in \mathbf{C}_\infty(\Gamma)$ are two consecutive clusters of mass at least k , then the interval $[\omega(\mathcal{C}_i) + 1, \alpha(\mathcal{C}_j) - 1]$ is a porous medium of level $(k - 1)$ with respect to Γ .*

Proof. It follows at once from the construction of $\mathbf{C}_{\ell-1}$ and \mathbf{C}_∞ . \square

Lemma 5.4. (Descending decomposition) *Each cluster $\mathcal{C} \in \cup_\ell \mathbf{C}_{\ell,\ell}(\Gamma)$ of mass $m \geq 2$ has the following representation: there exists an increasing sequence of integers*

$$\alpha(\mathcal{C}) = f_1 < g_1 < f_2 < g_2 < \cdots < f_v < g_v \leq \omega(\mathcal{C}) - 1,$$

so that for each $1 \leq s \leq v$, the partition $\mathbf{C}_\infty(\Gamma_{[f_s, g_s]})$ consists of a unique cluster, denoted by $\tilde{\mathcal{C}}_s$, with $\alpha(\tilde{\mathcal{C}}_s) = f_s$ and $\omega(\tilde{\mathcal{C}}_s) = g_s$, and the following holds:

- 1) $m(\tilde{\mathcal{C}}_1) = m - 1$, $m(\tilde{\mathcal{C}}_s) = \tilde{m}_s$ for $2 \leq s \leq v$, where $m - 1 \equiv \tilde{m}_1 > \tilde{m}_2 > \cdots > \tilde{m}_v$.
- 2) *the intervals $[g_{s-1} + 1, f_s - 1]$ are porous media of level \tilde{m}_s with respect to Γ , $2 \leq s \leq v$, and*

$$L^{\tilde{m}_s} \leq f_s - g_{s-1} \leq L^{\tilde{m}_s+1}; \quad (5.1)$$

$$\omega(\mathcal{C}) - L < g_v, \quad [g_v + 1, \omega(\mathcal{C}) - 1] \cap \Gamma = \emptyset. \quad (5.2)$$

Proof. Observe that the statement is obvious for clusters of level 1 and mass $m \geq 2$, in which case $v = 1$. We therefore consider clusters of level at least 2. The proof uses induction on the mass. Assuming the statement to be true for every cluster of mass at most m we prove it for every cluster of mass $m + 1$. Fix $\mathcal{C} \in \cup_\ell \mathbf{C}_{\ell,\ell}(\Gamma)$, such that $m(\mathcal{C}) = m + 1$. We split the proof in two sub-cases.

Case $\ell \equiv \ell(\mathcal{C}) = m$. In this case it follows from Lemma 5.1 that \mathcal{C} is formed as an m -run of only two constituents, \mathcal{C}_1 and \mathcal{C}_2 , with $m(\mathcal{C}_1) = m(\mathcal{C}_2) = m$, and we take $f_1 = \alpha(\mathcal{C})$ and $g_1 = \omega(\mathcal{C}_1)$. By Lemma 5.3 we have that $[\omega(\mathcal{C}_1) + 1, \alpha(\mathcal{C}_2) - 1]$ is porous media of level $m - 1$ and from the definition of the run we have that $L^{m-1} \leq \alpha(\mathcal{C}_2) - \omega(\mathcal{C}_1) < L^m$. On the other hand from the induction assumption we know that there exists a sequence of integers

$$\alpha(\mathcal{C}_2) = f'_1 < g'_1 < f'_2 < g'_2 < \cdots < f'_{v'} < g'_{v'} \leq \omega(\mathcal{C}_2) - 1,$$

such that for each $1 \leq s \leq v'$ the partition $\mathbf{C}_\infty(\Gamma_{[f'_s, g'_s]})$ consists of unique cluster, denoted by $\tilde{\mathcal{C}}'_s$ with $\alpha(\tilde{\mathcal{C}}'_s) = f'_s$ and $\omega(\tilde{\mathcal{C}}'_s) = g'_s$, with

$$m(\tilde{\mathcal{C}}'_1) = m - 1, \quad \text{and} \quad m(\tilde{\mathcal{C}}'_s) = \tilde{m}'_s, \quad 2 \leq s \leq v',$$

and the intervals $[g_{s-1} + 1, f_s - 1]$ are porous media with respect to Γ of level \tilde{m}'_s , $2 \leq s \leq v'$, and

$$L^{\tilde{m}'_s} \leq f_s - g_{s-1} \leq L^{\tilde{m}'_s+1}. \quad (5.3)$$

Taking $f_s = f'_{s-1}$ and $g_s = g'_{s-1}$, $2 \leq s \leq v'$, we get the desired representation of \mathcal{C} .

Case $2 \leq \ell \equiv \ell(\mathcal{C}) < m$. In this case \mathcal{C} is formed as an ℓ -run of r constituents $\mathcal{C}_1, \dots, \mathcal{C}_r$, $2 \leq r \leq m - \ell + 2$, with $m(\mathcal{C}_i) \geq \ell$, $1 \leq i \leq r$, and so $\mathbf{C}_\infty(\Gamma_{[\alpha(\mathcal{C}_1), \omega(\mathcal{C}_{r-1})]})$ consists of a unique cluster which we denote by $\hat{\mathcal{C}}$.

If $m(\hat{\mathcal{C}}) = m$, we see from (2.11) that $m(\mathcal{C}_r) = \ell < m$. In this case we set $f_1 = \omega(\mathcal{C}_{r-1})$, and using the inductive assumption for \mathcal{C}_r , we complete the representation as in the previous case.

If $m(\widehat{\mathcal{C}}) < m$, we have that $\ell + 1 \leq m(\mathcal{C}_r) = m - m(\widehat{\mathcal{C}}) + \ell \leq m$. By the inductive assumption applied to \mathcal{C}_r as the unique element of $\mathbf{C}_\infty(\Gamma_{[\alpha(\mathcal{C}_r), \omega(\mathcal{C}_r)]})$ there are integers

$$\alpha(\mathcal{C}_r) = \widetilde{f}_1 < \widetilde{g}_1 < \widetilde{f}_2 < \widetilde{g}_2 < \cdots < \widetilde{f}_{\widetilde{v}} < \widetilde{g}_{\widetilde{v}} \leq \omega(\mathcal{C}_r) - 1$$

for which properties 1)–2) of the lemma hold. Moreover, the unique cluster $\widetilde{\mathcal{C}}_1$ of $\mathbf{C}_\infty(\Gamma_{[\widetilde{f}_1, \widetilde{g}_1]})$ has mass $m - m(\widehat{\mathcal{C}}) + \ell - 1 \geq \ell$. In the configuration $\Gamma_{[\alpha(\mathcal{C}_1), \widetilde{g}_1]}$ the clusters $\mathcal{C}_1, \dots, \mathcal{C}_{r-1}$ and $\widetilde{\mathcal{C}}_1$ will form an ℓ -run, producing a cluster of mass m . Therefore, taking $f_s = \widetilde{f}_s$, $s \geq 2$ and $g_s = \widetilde{g}_s$, $1 \leq s \leq \widetilde{v}$, we get the desired representation of \mathcal{C} . \square

Definition 5.5. (Itinerary of a bad layer) *In the notation of the previous lemma, the sequence $\{\widetilde{m}_s\}_{s=1}^v$ will be called the itinerary of the descending decomposition.*

It follows from the construction that for any $1 \leq k \leq m - 1$ one can find $i_k \leq i'_k$ so that $B(m, \ell) = \cup_{i_k \leq s \leq i'_k} H_s^k$, and if $\ell < k$ we have $i_k = i'_k$. In particular it exists j so that $B(m, \ell) = H_j^{m-1}$. (For $\ell = 0$ this is a single bad line, and $m = 1$.)

Notice that if $k > \ell - 1$ in the above definition, then it is always the case that $i' = i$; the case $i' = i \pm 1$ may occur only if $k = \ell - 1$.

Definition 5.6. (Zones and tunnels) *If $S_{(i, i_k-1)}^k, \widehat{S}_{(i', i'_k+1)}^k$ form a matching pair with respect to $B(m, \ell)$, we set*

$$\mathcal{Z}(S_{(i, i_k-1)}^k, \widehat{S}_{(i', i'_k+1)}^k) = \left[(cL)^k \left(\frac{i - L^{1/2}}{2} \right), (cL)^k \left(\frac{i + L^{1/2}}{2} \right) \right] \times \mathcal{H}_j^{m-1},$$

which will be called the zone associated to $S_{(i, i_k-1)}^k$ and $\widehat{S}_{(i', i'_k+1)}^k$.

For $k \geq 1$ and if $i = i'$, we set

$$T(S_{(i, i_k-1)}^k, \widehat{S}_{(i', i'_k+1)}^k) = \left[\frac{i-1}{2}(cL)^k, \frac{i+1}{2}(cL)^k \right] \times \mathcal{H}_j^{m-1},$$

which we call the tunnel associated to $S_{(i, i_k-1)}^k$ and $\widehat{S}_{(i', i'_k+1)}^k$.

If $|i' - i| = 1$, the tunnel associated with $S_{(i, i_k-1)}^k$ and $\widehat{S}_{(i', i'_k+1)}^k$ is defined in the following way:

$$T(S_{(i, i_k-1)}^k, \widehat{S}_{(i', i'_k+1)}^k) = \left[\frac{i \wedge i'}{2}(cL)^k, \frac{i \vee i'}{2}(cL)^k \right] \times \mathcal{H}_j^{m-1}.$$

And finally, when $k = 0$ we set

$$T((i, i_0 - 1), (i', i'_0 + 1)) = \begin{cases} [i - 1, i] \times \mathcal{H}_j^{m-1} & \text{if } i = i' \\ [i \wedge i', i \vee i'] \times \mathcal{H}_j^{m-1} & \text{if } |i - i'| = 1. \end{cases}$$

Definition 5.7. (Vertical sequences) *A collection of k -sites $\{S_{(u_s, v_s)}^k\}_{s=1}^r$, with $v_{n+1} = v_n + 1$, $n = 1, \dots, r - 1$, is called a vertical sequence if $|u_1 - u_s| \leq 1$ for all $1 < s < r$.*

Definition 5.8. We say that the k -site $S_{(u_2, v_2)}^k$ lies above $S_{(u_1, v_1)}^k$, or, equivalently, $S_{(u_1, v_1)}^k$ lies below $S_{(u_2, v_2)}^k$, if $v_1 < v_2$, and $|u_1 - u_2| \leq 1$.

We will use the above definition also in the case of sequences of reversed sites.

Definition 5.9. (Separated pairs) Two matching pairs $S_{(i, i_k-1)}^k, \widehat{S}_{(i', i'_k+1)}^k$ and $S_{(j, i_k-1)}^k, \widehat{S}_{(j', i'_k+1)}^k$ are said to be separated if $|j - i| \geq 2L^{1/2}$.

Notice that if two matching pairs are separated, their corresponding zones do not intersect.

Notation. For an horizontal segment $I = \{(x, y) \in \widetilde{\mathbb{Z}}_+^2 : a \leq x \leq b\}$ we denote

$$\begin{aligned} I_\gamma &= \{(x, y) \in \widetilde{\mathbb{Z}}_+^2 : a + \lfloor (b-a)/12 \rfloor + 1 \leq x \leq a + 2\lfloor (b-a)/12 \rfloor - 1\}, \\ I_\Gamma &= \{(x, y) \in \widetilde{\mathbb{Z}}_+^2 : a + 10\lfloor (b-a)/12 \rfloor + 1 \leq x \leq a + 11\lfloor (b-a)/12 \rfloor - 1\}. \end{aligned}$$

Definition 5.10. An horizontal segment $I = \{(x, y) \in \widetilde{\mathbb{Z}}_+^2 : a \leq x \leq b\}$ with $b - a = (cL)^k$ is called k -segment either if it is contained in some good k -site S^k , or if there is a bad layer $B(m, \ell)$, $\ell < k$, and two good k -sites S^k, \widehat{S}^k forming a matching pair with respect to $B(m, \ell)$ such that $I \subset T_{(S^k, \widehat{S}^k)}$. We denote a k -segment I by I^k .

Definition 5.11. (hierarchical k -set) Given a k -segment I^k , a collection $\overline{I^k}$ of ℓ -segments $\{I_j^\ell\}_j$, $\ell = 0, \dots, k$, contained in I^k is a hierarchical k -set associated with I^k if:

- i) I^k is the unique k -segment in the collection;
- ii) $I_j^\ell \cap I_{j'}^\ell = \emptyset$ if $j \neq j'$, for any $\ell \in \{0, \dots, k-1\}$;
- iii) for $k \geq 1$ and $\ell \in \{1, \dots, k\}$, each interval $(I_j^\ell)_\gamma$ and $(I_j^\ell)_\Gamma$ contains at least $\frac{1}{12}\rho cL$ $(\ell-1)$ -segments in $\overline{I^k}$.

When $k = 0$ we simply have $\overline{I^0} = \{I^0\} = \{S^0\}$ for a 0-site S^0 and we identify $\overline{I^0}$ with I^0 . For $k \geq 1$ and having fixed I^k , it is convenient to label the elements of $\overline{I^k}$: going down, from $\ell = k-1$ to $\ell = 0$, we label all ℓ -segments contained in each $I^{\ell+1}$ from left to right, starting the numbering within each $I^{\ell+1}$ every time from 1. Proceeding in this way, we have a multi-index $\mu_{\langle k, \ell \rangle} = \langle \mu_{k-1}, \mu_{k-2}, \dots, \mu_\ell \rangle$ which indicates the “genealogical tree” down to scale ℓ . We denote the corresponding ℓ -segment with this index by $I_{\mu_{\langle k, \ell \rangle}}^\ell$. We shall also use the following convention. If $\mu_\ell = j$, we will write:

$$\langle \mu_{k-1}, \dots, \mu_{\ell+1}, j \rangle = \langle \mu_{\langle k, \ell+1 \rangle}, j \rangle. \quad (5.4)$$

Definition 5.12. (a) For any type 2 good k -site S^k , $k \geq 1$, $\Psi^k(S^k)$ denotes its top 0-layer. Analogously, if \widehat{S}^k is a good reverse k -site of type 2, $\Upsilon^k(\widehat{S}^k)$ denotes its bottom 0-layer. When $k = 0$, we set $\Psi^0(S^0) = S^0$ and $\Upsilon^0(\widehat{S}^0) = \widehat{S}^0$.

(b) If the k -sites S^k and \widehat{S}^k form a matching pair with respect to $B(m, \ell)$ in the sense of Definition 4.6, and $\ell-1 \leq k \leq m-1$, we say that $\Psi^k(S^k)$ and $\Upsilon^k(\widehat{S}^k)$ also form a matching pair;

(c) Two hierarchical k -sets $\overline{\Psi^k}$ and $\overline{\Upsilon^k}$ whose k -segments Ψ^k and Υ^k form a matching pair with respect to $B(m, \ell)$, with $\ell - 1 \leq k \leq m - 1$, are also called a matching pair with respect to $B(m, \ell)$.

For the proof in Section 6 we shall use the following hierarchical k -sets: let S^k be a good k -site with dense kernel. In this case, there will be at least $\frac{1}{12}\rho cL$ $(k - 1)$ -sites in $D_l^K(S^k)$ and $D_r^K(S^k)$ respectively, and each of them will have dense kernel. The same happens at all scales down 0. The top 0-layers of the kernel of these dense kernel sites at all scales form a hierarchical k -set, which we denote as $\overline{\Psi^k}(S^k)$. The analogous hierarchical k -set for a reverse k -site \widehat{S}^k we will denote by $\overline{\Upsilon^k}(\widehat{S}^k)$. We shall use this in the case when S^k is of type 2, lying immediately below a bad layer of mass larger than k (so that S^k coincides with its kernel) or when it contains a bad layer of mass k (type 1).

Notation. It will be convenient to single out the class of level 1 bad layers $B(m, 1)$ consisting of m consecutive bad lines. We call such bad layers “monolithic”, and refer to them as bad 1_M -layers. In this case we write $B(m, 1_M)$.

The following concept of *chaining* plays an important role in the proof in Section ?? . We split it into two definitions, for “large” and “small” hierarchical sets, where large or small has to do with the level of the bad layer, as made precise below.

Definition 5.13. (“Large” chained hierarchical k -sets) *Let $B(m, \ell)$ be a bad layer of level ℓ and mass m . If $k \in \{\ell - 1, \dots, m - 1\}$, two hierarchical k -sets $\overline{\Psi^k}$ and $\overline{\Upsilon^k}$ forming a matching pair with respect to $B(m, \ell)$ are said to be chained through $B(m, \ell)$ if the following holds:*

The case $k = 0$. *In this situation $\ell = 1$, and we distinguish the 1_M -layers from the remaining $B(m, 1)$ layers.*

- *1_M -layer.* We say that $\overline{\Psi^0}$ and $\overline{\Upsilon^0}$ are chained if there exists an open vertical path of 0-sites from a nearest neighbor of Ψ^0 to nearest neighbor of Υ^0 ;
- *Non-monolithic layer.* In this case $B(m, 1)$ is formed by m bad lines grouped into $r > 1$ 1_M -layers³, separated among themselves by at most $L - 1$ good lines. We denote these parts by $B_v(m_v, 1_M)$, $1 \leq v \leq r$, where m_v is the number of bad lines it contains. We say that $\overline{\Psi^0}$ and $\overline{\Upsilon^0}$ are chained through $B(m, 1)$ if there exist a vertical sequence of hierarchical 0-sets

$$\widehat{S}^0(1), S^0(2), \widehat{S}^0(2), \dots, S^0(r),$$

such that

- a) *all sites $S^0(v)$, $v = 2, \dots, r$, and $\widehat{S}^0(s)$, $v = 1, \dots, r - 1$, are passable;*
- b) *the sites $S^0(v)$ and $\widehat{S}^0(v)$, $v = 2, \dots, r - 1$, form a matching pair with respect to $B_v(m_v, 1_M)$;*
- c) *Ψ^0 and $S^0(1)$ are chained through $B_1(m_1, 1_M)$; $S^0(v)$ and $\widehat{S}^0(v)$ are chained through $B_v(m_v, 1_M)$, for each $v = 2, \dots, r - 1$; $S^0(r)$ and Υ^0 are chained through $B_r(m_r, 1_M)$.*

³this is slight abuse of our previous notation

d) each pair of sites $\widehat{S}^0(s)$ and $S^0(s+1)$, $1 \leq s \leq r-1$, is connected by an open path of 0-sites lying within $\mathcal{Z}(S_{(i,i_k-1)}^0, \widehat{S}_{(i',i'_k+1)}^0)$.

The case $k \geq 1$. Again we distinguish two cases:

- $k \geq \ell$. Since $\rho > 1/2$, letting $\widehat{\rho} = \rho - 1/2$, the definition of hierarchical set implies the existence of at least $\widehat{\rho}_6^c L$ matching pairs $\overline{\Psi}^{k-1}, \overline{\Upsilon}^{k-1}$ with respect to $B(m, \ell)$, with $\overline{\Psi}^{k-1} \subset \overline{\Psi}^k$ and $\overline{\Upsilon}^{k-1} \subset \overline{\Upsilon}^k$. Let \mathcal{M} be the set formed by the first (from left to right) $\widehat{\rho}_6^c L^{1/2}$ such pairs which are separated. We say that $\overline{\Psi}^k$ and $\overline{\Upsilon}^k$ are chained if at least one matching pair in \mathcal{M} is chained through $B(m, \ell)$.
- $k = \ell - 1$. Assume that $B(m, \ell)$ has $r > 1$ constituents of masses m_v and levels $\ell_v < \ell$, hereby denoted as $B_v(m_v, \ell_v)$, $v = 1, \dots, r$. We say that $\overline{\Psi}^k$ and $\overline{\Upsilon}^k$ are chained if there exist a vertical sequence of good k -sites

$$\widehat{S}^k(1), S^k(2), \widehat{S}^k(2), \dots, S^k(r),$$

such that

- a) For each $1 \leq v \leq r-1$, $S(\widehat{S}^k(v))$ and $S^k(v+1)$ are connected by a passable k -path, lying entirely in $\mathcal{Z}(S_{(i,i_k-1)}^k, \widehat{S}_{(i',i'_k+1)}^k)$. (In particular, $S^k(v)$ has s -dense kernel, $v = 2, \dots, r$.)
- b) $\widehat{S}^k(v)$ has c -dense kernel (reversed), $v = 1, \dots, r-1$.
- c) $S^k(v)$ and $\widehat{S}^k(v)$ form a matching pair which respect to $B_v(m_v, \ell_v)$, $v = 2, \dots, r-1$.
- d) $\overline{\Psi}^k$ and $\overline{\Upsilon}^k(\widehat{S}^k(1))$ are chained through $B_1(m_1, \ell_1)$; $\overline{\Psi}^k(S^k(v))$ and $\overline{\Upsilon}^k(\widehat{S}^k(v))$ are chained through $B_v(m_v, \ell_v)$, $v = 2, \dots, r-1$; and finally $\overline{\Psi}^k(S^k(r))$ and $\overline{\Upsilon}^k$ are chained through $B_r(m_r, \ell_r)$.

The connections by open oriented paths that are examined using the iterative procedure just defined will be called restricted.

Notation. For easiness of notation we shall write $J = \widehat{\rho}_6^c L^{1/2}$.

Remarks.

- a) Let $B(m, \ell)$ be a bad layer of mass m and level ℓ through which a matching pair of hierarchical k -sets $\overline{\Psi}^k$ and $\overline{\Upsilon}^k$ is chained. If $k \geq \ell$, there exists an open oriented (restricted) path of 0-sites crossing $B(m, \ell)$ and lying in $T(S_{(i,i_k-1)}^k, \widehat{S}_{(i',i'_k+1)}^k)$; if $k = \ell - 1$ such a path exists in $\mathcal{Z}(S_{(i,i_k-1)}^k, \widehat{S}_{(i',i'_k+1)}^k)$.
- b) Notice that in Definition 5.13, for each $j \in \{0, \dots, k-1\}$, we examine at each step (according to the set \mathcal{M} in Definition 5.13) exactly J j -segments within each checked $j+1$ -segment in $\overline{\Upsilon}^k$, and similarly for $\overline{\Psi}^k$; each checked to be connected to different j -segments within $B(m, \ell)$. The algorithm for selecting \mathcal{M} at each smaller scale depends (except in the trivial case of layers $B(m, 1_M)$) on what happens within $B(m, \ell)$ as explained therein. With some abuse of notation we call $\mathcal{M}(\overline{\Upsilon}^k)$ and similarly $\mathcal{M}(\overline{\Psi}^k)$ the collection of these checked segments at all scales (J at each scale).

c) The estimates in the next section become easier to formulate once the number of j -segments to be examined within a $j+1$ -segment is fixed at all times. But it does not depend on the exact algorithm to define the set \mathcal{M} and for the construction in Section 6 this will be slightly different then the one used in the above definition, though we shall use the same notation.

Definition 5.14. (“Small” chained hierarchical sets) *Let $\ell \geq 2$, $k \in \{\ell-1, \dots, m-1\}$, and let $B(m, \ell)$ be a bad layer of mass m and level ℓ for which $\overline{\Psi}^k$ and $\overline{\Upsilon}^k$ form a matching pair of hierarchical k -sets, assumed to be chained according to Definition 5.13.*

a) *We say that a 0-site $\Upsilon_{\mu_{(k,0)}}^0 \in \mathcal{M}(\overline{\Upsilon}^k)$ is chained to $\overline{\Psi}^k$, if there exists a 0-site $\Psi_{\mu_{(k,0)}}^0 \in \mathcal{M}(\overline{\Psi}^k)$, and a restricted open path from a nearest neighbor (from above) of $\Psi_{\mu_{(k,0)}}^0$ to a nearest neighbor of $\Upsilon_{\mu_{(k,0)}}^0$ from below.*

b) *We say that the r -segment $\Upsilon_{\mu_{(k,r)}}^r \in \mathcal{M}(\overline{\Upsilon}^k)$, $0 < r < \ell-1$, is chained to $\overline{\Psi}^k$ if it contains an $(r-1)$ -segment $\Upsilon_{\mu_{(k,r-1)}}^{r-1} \in \mathcal{M}(\overline{\Upsilon}^k)$ which is chained to $\overline{\Psi}^k$.*

Remark. A 0-path as in a) above will be open, oriented, and will lie entirely in the tunnel $T(\Upsilon^k, \Psi^k)$ when $k \geq \ell$, and for $k = \ell-1$ it will lie in $Z(\Upsilon^{\ell-1}, \Psi^{\ell-1})$.⁴

Definition 5.15. (chained k -sites) *Let $B(m, \ell)$ be a bad layer of level ℓ and mass m , and $k \in \{\ell-1, \dots, m-1\}$. Two k -sites S^k and \hat{S}^k forming a matching pair with respect to $B(m, \ell)$ are said to be chained through $B(m, \ell)$, if the corresponding hierarchical k -sets $\overline{\Psi}^k(S^k)$ and $\overline{\Upsilon}^k(\hat{S}^k)$ are chained through $B(m, \ell)$.*

Notation. The event of two hierarchical k -sets $\overline{\Psi}^k$, $\overline{\Upsilon}^k$, or analogously two k -sites S^k , \hat{S}^k being chained through $B(m, \ell)$ is denoted by

$$\overline{\Psi}^k \overset{\rightsquigarrow}{B(m, \ell)} \overline{\Upsilon}^k \quad (5.5)$$

and respectively

$$S^k \overset{\rightsquigarrow}{B(m, \ell)} \hat{S}^k. \quad (5.6)$$

6. CONCLUSION OF THE PROOF OF THEOREM 4.11

Notation. Let $\varkappa > 0$ to be fixed later. We recursively define for all $m \geq 1$:

$$\begin{aligned} p_{0,m} &:= p_B^m p_G^{\varkappa(m-1)}, \\ p_{k,m} &:= (1 - (1 - p_{k-1,m})^J) p_k^{\varkappa(m-k-1)}, \quad 1 \leq k \leq m-1, \\ p_{m,m} &:= 1 - (1 - p_{m-1,m})^J, \end{aligned} \quad (6.1)$$

⁴Defined analogously to $T(S, \hat{S})$ and $Z(S, \hat{S})$.

where $J = \hat{\rho}_6^c L^{1/2}$, $\hat{\rho} = \rho - 1/2$ as in Definition 5.13, $p_k = 1 - q_k$, $q_k = q_0^{k+1}$, $p_0 = p_G$, $q_0 = 1 - p_G$, as in Theorem 4.11.

Recalling the statement of Theorem 4.11 and what has been proven in Section 4, it remains to verify that (b_{m+1}) follows from $(a_j), (b_j), (c_j), (d_j)$ for all $j \leq m$ and (a_{m+1}) . To get such estimates we need a more detailed analysis, as developed in the last section. For $m \geq 1$, let $p_{k,m}$ be given as above, and set:

$(b_m)'$ For every bad layer $B(m, \ell)$ of mass m (any level ℓ), every $j \in \{\ell - 1, \dots, m - 1\}$ and every hierarchical j -sets $\overline{\Psi}^j, \overline{\Upsilon}^j$ that form a matching pair with respect to $B(m, \ell)$,

$$P(\overline{\Psi}^j \overset{\sim}{\leftarrow}_{B(m, \ell)} \overline{\Upsilon}^j) \geq p_{j,m}. \quad (6.2)$$

For $m \geq 2$:

$(b_m)''$ For every $B(m, \ell)$, j , $\overline{\Psi}^j, \overline{\Upsilon}^j$ as in $(b_m)'$, and every $s \in \{0, \dots, j - 1\}$, the distribution of the number of $\Upsilon_{\langle \mu_{(j,s+1)}, i \rangle}^s \in \mathcal{M}(\overline{\Upsilon}^j)$ that are chained to $\overline{\Psi}^j$, conditioned on $\Upsilon_{\mu_{(j,s+1)}}^{s+1}$ being chained to $\overline{\Psi}^j$, is stochastically larger than $F_{p_{s,m}}$, where F_p denotes the distribution of a Binomial random variable with J trials and success probability p , conditioned to have at least one success. That is,

$$|\{i: \Upsilon_{\langle \mu_{(j,s+1)}, i \rangle}^s \in \mathcal{M}(\overline{\Upsilon}^j): \Upsilon_{\langle \mu_{(j,s+1)}, i \rangle}^s \text{ chained to } \overline{\Psi}^j\}| \Big| [\Upsilon_{\mu_{(j,s+1)}}^{s+1} \text{ chained to } \overline{\Psi}^j] \succeq F_{p_{s,m}} \quad (6.3)$$

with \succeq standing for stochastically larger in the usual sense.

Theorem 6.1. *The properties $(a_m), (d_m)$ of Theorem 4.11 and the above $(b_m)'$ hold for every $m \geq 1$; $(b_m)''$ holds for every $m \geq 2$.*

Proof. The proof is by induction in m .

Initial step. $(b_1)'$ follows directly from the definitions, and $(a_1), (d_1)$ have already been verified in Section 4. $(b_2)''$ is also trivially verified.

Induction step. We first establish $(b_{m+1})'$. Throughout the proof we use the descending decomposition representation of the bad layer $B(m + 1, \ell)$; we also construct a class of particularly chosen hierarchical sets that will play a role in the induction.

Let $(\overline{\Psi}^m, \overline{\Upsilon}^m)$ form a matching pair with respect to a bad layer $B(m + 1, \ell)$, and let $\{\tilde{m}_s\}_{s=1}^v$ denote the itinerary of the descending decomposition of $B(m + 1, \ell)$, with $\{\tilde{\mathcal{C}}_s\}_{s=1}^v$ its corresponding clusters, and $\{B_{\tilde{\mathcal{C}}_s}\}_{s=1}^v$ the corresponding bad layers. The interval between any two consecutive clusters $\tilde{\mathcal{C}}_s$ and $\tilde{\mathcal{C}}_{s+1}$ is always porous media of level \tilde{m}_{s+1} (see Lemma 5.4).

An *entrance set* $\overline{\Psi}^m(s)$, $s = 2, \dots, v$, will be a suitable hierarchical m -set located at the 0-layer just below $B_{\tilde{\mathcal{C}}_s}$, and an *exit set* $\overline{\Upsilon}^m(s)$, $s = 1, \dots, v$, a suitable hierarchical m -set located at the 0-layer just above $B_{\tilde{\mathcal{C}}_s}$ for $s = 1, \dots, v - 1$, with $\overline{\Upsilon}^m(v)$ located at the 0-layer just above $B_{\tilde{\mathcal{C}}_v}$, or at its last 0-layer, according to $g_v < \omega(\mathcal{C}) - 1$ or $g_v = \omega(\mathcal{C}) - 1$ (Lemma 5.4).

Large segments of exit and entrance sets. For each $s = 1, \dots, v-1$, the m -segment $\Upsilon^m(s)$ and all j -segments $\Upsilon_{\mu_{\langle m, j \rangle}}^j(s)$, $\tilde{m}_{s+1} \leq j < m$, in $\overline{\Upsilon^m}(s)$, are obtained by taking the corresponding segments Υ^m and $\Upsilon_{\mu_{\langle m, j \rangle}}^j$ and projecting them vertically on the 0-layer located just above $B_{\tilde{\mathcal{C}}_s}$, and then by taking as $\Upsilon_{\mu_{\langle m, j \rangle}}^j(s)$ a j -segment which intersects this projection: when there are two such j -segments, to avoid ambiguities we take the one which intersects the left half of the projection. For $s = v$ the only difference is that when $g_v = \omega(\mathcal{C}) - 1$ the segments will be located at the last 0-layer of $B_{\tilde{\mathcal{C}}_v}$.

For the entrance sets $\overline{\Psi^m}(s)$ with $s = 2, \dots, v$ we proceed in the same way: the m -segment $\Psi^m(s)$ and all j -segments $\Psi_{\mu_{\langle m, j \rangle}}^j(s)$, $\tilde{m}_s \leq j < m$, in $\overline{\Psi^m}(s)$, are obtained by taking the corresponding segments Υ^m , and $\Upsilon_{\mu_{\langle m, j \rangle}}^j$ and projecting them vertically on the 0-layer located just below $B_{\tilde{\mathcal{C}}_s}$, with the same selection rule as above in case there are two such j -segments.

Construction of the exit sets $\overline{\Upsilon^m}(s)$. Consider first the case $1 \leq s < v$. To continue the construction of the j -segments at scales smaller than \tilde{m}_{s+1} , we consider, for each already defined \tilde{m}_{s+1} -segment of this collection, the reversed \tilde{m}_{s+1} -site for which this segment is the last 0-layer, i.e. $\hat{S}^{\tilde{m}_{s+1}}$ such that $\Upsilon(\hat{S}^{\tilde{m}_{s+1}}) = \Upsilon_{\mu_{\langle m, \tilde{m}_{s+1} \rangle}}^{\tilde{m}_{s+1}}(s)$, and check if this site has c -(reverse) dense kernel. If the answer is affirmative, we take $\Upsilon_{\mu_{\langle m, j \rangle}}^j(1)$, $j = 0, \dots, \tilde{m}_{s+1} - 1$ as the bottom 0-layers of the reverse dense kernel sites of $\hat{S}^{\tilde{m}_{s+1}}$, or in other words all the scales from \tilde{m}_{s+1} down to zero correspond to $\overline{\Upsilon^{\tilde{m}_{s+1}}}(\hat{S}^{\tilde{m}_{s+1}})$. These are called “compatible” segments. For those sites $\hat{S}^{\tilde{m}_{s+1}}$ that do not have c -(reverse) dense kernel, we select the $\Upsilon_{\mu_{\langle m, j \rangle}}^j(s)$, $j = 0, \dots, \tilde{m}_{s+1} - 1$ in an arbitrary way among the correspondent sub-segments of bottom 0-layers of the site. We call such choice of segments “incompatible” with the process. Only compatible segments will play a role in the construction.

In the case $s = v$ we make essentially the same construction, as if $\tilde{m}_{s+1} = 0$, with the difference that when $\omega(B(m+1, \ell)) = \omega(\tilde{\mathcal{C}}_v) + 1$ we locate the hierarchical set at the last 0-layer of $B_{\tilde{\mathcal{C}}_v}$.

Observe that the construction of $\overline{\Upsilon^m}(s)$ and the compatibility of its segments depend on Γ and on the occupation variables in between $B_{\tilde{\mathcal{C}}_s}$ and $B_{\tilde{\mathcal{C}}_{s+1}}$.

Step 1, part 1. We check if at least one among the pairs of hierarchical $(m-1)$ -sets $\overline{\Psi_{\mu_{\langle m, m-1 \rangle}}^{m-1}}$ and $\overline{\Upsilon_{\mu_{\langle m, m-1 \rangle}}^{m-1}}(1)$ is chained through $B_{\tilde{\mathcal{C}}_1}$. If so, we move to the next item; otherwise we stop the procedure and say that $\overline{\Psi^m}$ and $\overline{\Upsilon^m}$ are not chained through $B(m+1, \ell)$.

Step 1, part 2. (Zooming) For each pair of hierarchical $(m-1)$ -sets $\overline{\Psi_{\mu_{\langle m, m-1 \rangle}}^{m-1}}$ and $\overline{\Upsilon_{\mu_{\langle m, m-1 \rangle}}^{m-1}}(1)$ chained through $B_{\tilde{\mathcal{C}}_1}$ we select all multi-indices $\mu_{\langle m, \tilde{m}_2 \rangle}$ and the corresponding \tilde{m}_2 -segments $\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}(1)$, which are compatible and from which there exists an open oriented 0-level path that connects to $\Psi_{\mu_{\langle m, m-1 \rangle}}^{m-1}$ through $B_{\tilde{\mathcal{C}}_1}$.

Step 1, part 3. (Transfer) For the \tilde{m}_2 -segments $\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}(1)$ selected in the previous item, we first check whether

(i) the forward site $S(\hat{S}_{\mathbf{x}}^{\tilde{m}_2})$ is c -(forward) passable.

If the answer is positive, it implies that at least one of the seeds of $Q_l(S(\hat{S}_{\mathbf{x}}^{\tilde{m}_2}))$ or $Q_r(S(\hat{S}_{\mathbf{x}}^{\tilde{m}_2}))$ is also connected to Ψ^{m-1} (we may call it “active”). This gives us a way of completing the construction of the hierarchical set $\overline{\Psi^m}(2)$ at scales smaller than \tilde{m}_2 :

Construction of the entrance set $\overline{\Psi^m}(2)$. Take $S_{\mathbf{x}'}^{\tilde{m}_2}$ such that $\Psi(S_{\mathbf{x}'}^{\tilde{m}_2}) = \Psi_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}(2)$, and check whether

(ii) there exists an oriented passable \tilde{m}_2 -path starting from the \tilde{m}_2 -site which is s -passable from the active seed of $S(\hat{S}_{\mathbf{x}}^{\tilde{m}_2})$ to $S_{\mathbf{x}'}^{\tilde{m}_2}$, and entirely contained in $\mathcal{Z}(S(\hat{S}_{\mathbf{x}}^{\tilde{m}_2}), S_{\mathbf{x}'}^{\tilde{m}_2})$.

If the answer to (i) and (ii) is positive we say that $\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}}(1)$ and $\overline{\Psi_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}}(2)$ are “active”. Otherwise the procedure of building connection from $\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}(1)$ is stopped. This completes Step 1.

Remark 6.2. Notice that $\overline{\Psi^m}(2)$ lies just below $B_{\tilde{C}_2}$, but a positive answer to (i) and (ii) above, besides guaranteeing the connection of $\overline{\Psi^m}(2)$ to the corresponding $\Upsilon^{\tilde{m}_2}(1)$ (and therefore to Ψ^{m-1} by force of the previous sub-step) also gives connection by open oriented path of 0-sites to suitable sites at the top 0-layer of $B_{\tilde{C}_2}$ (according to the definition of passability at the scale \tilde{m}_2), which then implies the existence of an open path to a 0-site in $B_{\tilde{C}_2}$ which is nearest neighbor of a corresponding \tilde{m}_2 -segment $\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}(2)$. The first part does not depend on the occupation variables in $B_{\tilde{C}_2}$, and one might find convenient to think of the event in (ii) as the intersection of these two conditions involving disjoint sets of 0-sites.

Step s , $1 < s \leq v$. Having determined the “active” $\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_s \rangle}}^{\tilde{m}_s}}(s-1)$ and $\overline{\Psi_{\mu_{\langle m, \tilde{m}_s \rangle}}^{\tilde{m}_s}}(s)$, the process continues only from the “compatible” corresponding sub-segments $\Upsilon_{\mu_{\langle m, \tilde{m}_{s+1} \rangle}}^{\tilde{m}_{s+1}}(s)$.

Sub-case $s < v$. The construction repeats what was done above for $s = 1$:

- We check if $\overline{\Psi_{\mu_{\langle m, \tilde{m}_s \rangle}}^{\tilde{m}_s}}(s)$ and $\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_s \rangle}}^{\tilde{m}_s}}(s)$ are chained through $B_{\tilde{C}_s}$;
- For each pair of hierarchical $(\tilde{m}_s - 1)$ -sets $\overline{\Psi_{\mu_{\langle m, \tilde{m}_{s-1} \rangle}}^{\tilde{m}_{s-1}}}(s)$ and $\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_{s-1} \rangle}}^{\tilde{m}_{s-1}}}(s)$ chained through $B_{\tilde{C}_s}$, we select all multi-indices $\mu_{\langle m, \tilde{m}_{s+1} \rangle}$ and corresponding \tilde{m}_{s+1} -segments $\Upsilon_{\mu_{\langle m, \tilde{m}_{s+1} \rangle}}^{\tilde{m}_{s+1}}(s)$ which are compatible and for which there exists a 0-level path (open, oriented) connecting them to $\Psi_{\mu_{\langle \tilde{m}_s, \tilde{m}_{s-1} \rangle}}^{\tilde{m}_{s-1}}$ through $B_{\tilde{C}_s}$.
- item 2, called “transfer”, and the construction of $\overline{\Psi^m}(s+1)$ both follow the same procedure as when $s = 1$, replacing \tilde{m}_2 by \tilde{m}_{s+1} . We then say that $\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_{s+1} \rangle}}^{\tilde{m}_{s+1}}}(s)$ and $\overline{\Psi_{\mu_{\langle m, \tilde{m}_{s+1} \rangle}}^{\tilde{m}_{s+1}}}(s+1)$ are active if the analogue of the previous (i)–(ii) both hold.

Sub-case $s = v$. This splits into two situations:

a) $\omega(B(m+1, \ell)) > \omega(\tilde{\mathcal{C}}_v) + 1$. In this case we act as if $\tilde{m}_{s+1} = 0$, i.e. we first of all perform “zooming” selecting all active elements down to 0 level, and repeat the “transfer” procedure.

b) $\omega(B(m+1, \ell)) = \omega(\tilde{\mathcal{C}}_v) + 1$. In this case we act as if $\tilde{m}_{s+1} = 0$, i.e. we first of all perform the “zooming” selecting all active elements down to 0 level, however the “transfer” procedure reduces to connecting over the last bad line of $B(m+1, \ell)$.

Estimates needed for the induction step.

In what follows, we will repeatedly use the following basic result on the standard oriented percolation model on $\tilde{\mathbb{Z}}_+^2$:

For $a \geq 1$ a large integer, consider the rectangle $R_a = ([0, a] \times [0, a^2]) \cap \tilde{\mathbb{Z}}_+^2$, and let $(x, 0)$, (y, a^2) be two points lying on the two horizontal faces, with $|x - a/2|, |y - a/2| \leq a/10$, and define the following event of vertical crossing:

$V(R_a) = [\text{there exists an open oriented path from } (x, 0) \text{ to } (y, a^2) \text{ lying entirely in } R_a]$

Then the following holds:

Lemma 6.3. *There exist $a_0 \geq 1, 0 < \tilde{p} < 1$ and $\varkappa' > 0$, such that for any $a \geq a_0$ and $p \geq \tilde{p}$ we have*

$$P_p(V(R_a)) \geq p^{\varkappa'}.$$

Proof. The proof of the above inequality is rather standard. We sketch it briefly: let us split the rectangle into a disjoint squares with sides a , and choose a large enough, with \tilde{p} close enough to 1 so that the probability of survival from a starting point (centered) in the first (from the bottom, say) square $a \times a$ is larger than $p^{\varkappa'}$, and given that the process survives in the square, at its upper boundary it is close to its asymptotic shape and asymptotic density. Then we repeatedly request survival and approximation to asymptotic density in the next $a - 1$ consecutive squares, starting from $a/(5\theta(p_0))$ centrally located points. The probability of each of such events is exponentially $\exp(-c_1 a)$, with constant $c_1 > 0$, and uniformly bounded away from 0, for p large. We easily get the desired result. \square

Remark 6.4. *The previous lemma is used in the part of the procedure called “transfer” above. It will be used at the various scales $k \leq m$, with $\tilde{p} = p_k$, and $a^2 \geq L$ which we may assume large enough so that the estimate applies.*

Let $(\overline{\Psi^m}, \overline{\Upsilon^m})$ be a matching pair with respect to $B(m+1, \ell)$ under consideration. By the induction assumption $(b_m)'$, we have that for each pair of indices $\mu_{\langle m, m-1 \rangle}$

$$P \left(\overline{\Upsilon_{\mu_{\langle m, m-1 \rangle}}^{m-1}}(1) \overset{B(\tilde{\mathcal{C}}_1)}{\longleftrightarrow} \overline{\Psi_{\mu_{\langle m, m-1 \rangle}}^{m-1}} \right) \geq p_{m-1, m}. \quad (6.4)$$

For a fixed family of hierarchical $(m-1)$ -sets, the events in (6.4) are (conditionally) independent, so that the distribution of the number of chained pairs, given that at least one of them is chained, is stochastically larger than $F_{p_{m-1,m}}$.

On the other hand, from the induction assumption $(b_m)''$ we have that for each $0 \leq j < m-1$ and each pair $\mu_{\langle m, j+1 \rangle}$

$$|\{i: \overline{\Upsilon_{\mu_{\langle m, j+1 \rangle}, i}^j}(1) \overset{\rightsquigarrow}{\underset{B(\tilde{\mathcal{C}}_1)}{\longleftrightarrow}} \overline{\Psi_{\mu_{\langle m, m-1 \rangle}}^{m-1}}\}| | [\overline{\Upsilon_{\mu_{\langle m, j+1 \rangle}}^{j+1}}(1) \overset{\rightsquigarrow}{\underset{B(\tilde{\mathcal{C}}_1)}{\longleftrightarrow}} \overline{\Psi_{\mu_{\langle m, m-1 \rangle}}^{m-1}}] \succeq F_{p_{j,m}}, \quad (6.5)$$

i.e. conditioned on $\overline{\Upsilon_{\mu_{\langle m, j+1 \rangle}}^{j+1}}(1)$ being chained to $\overline{\Psi_{\mu_{\langle m, m-1 \rangle}}^{m-1}}$, the number of indices i so that $\overline{\Upsilon_{\mu_{\langle m, j+1 \rangle}, i}^j}(1)$ is chained to $\overline{\Psi_{\mu_{\langle m, m-1 \rangle}}^{m-1}}$ is stochastically larger than $F_{p_{j,m}}$. We shall use (6.5) for j going down to $j = \tilde{m}_2$.

Assume $\tilde{m}_2 \geq 1$, i.e. $v \geq 2$. For each index $\mu_{\langle m, \tilde{m}_2 \rangle}$ which yields a chained set at all steps from $m-1$ down to \tilde{m}_2 one now checks the \tilde{m}_2 -set $\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}}$ is “compatible” and if the conditions (i) and (ii) described in the previous construction hold. Using the induction assumption, which guarantees the validity of conditions $(a_i) - (d_i)$ for all $i \leq m$. Applying this and Lemma 6.3, we get from (6.3), for each such index $\mu_{\langle m, \tilde{m}_2 \rangle}$:

$$P \left(\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}}(1) \text{ and } \overline{\Psi_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}}(2) \text{ are “active”} \right) \geq p_{\tilde{m}_2}^{\kappa} \quad (6.6)$$

where $\kappa = \kappa' + 2$ (the $+2$ appears since we need to check that the starting \tilde{m}_2 -site at the bottom has reverse c -dense kernel (compatible), and is forward c -passable). Using then (a_m) we get that for all the previous indices $\mu_{\langle m, \tilde{m}_2 \rangle}$ as above (for them we have $\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}}(1)$ is chained to $\overline{\Psi_{\mu_{\langle m, m-1 \rangle}}^{m-1}}$) one gets

$$P \left(\exists i: \overline{\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}, i}^{\tilde{m}_2-1}}(2) \overset{\rightsquigarrow}{\underset{B(\tilde{\mathcal{C}}_2)}{\longleftrightarrow}} \overline{\Psi_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2-1}}(2) \right) \geq p_{\tilde{m}_2, \tilde{m}_2}. \quad (6.7)$$

The event on the l.h.s. of (6.7) we naturally denote as

$$\left[\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}}(2) \overset{\rightsquigarrow}{\underset{B(\tilde{\mathcal{C}}_2)}{\longleftrightarrow}} \overline{\Psi_{\mu_{\langle m, \tilde{m}_2 \rangle}}^{\tilde{m}_2}}(2) \right]$$

and by the induction assumption we can write, analogously to (6.5), for each $j < \tilde{m}_2 - 1$:

$$|\{i: \overline{\Upsilon_{\mu_{\langle m, j+1 \rangle}, i}^j}(1) \overset{\rightsquigarrow}{\underset{B(\tilde{\mathcal{C}}_2)}{\longleftrightarrow}} \overline{\Psi_{\mu_{\langle m, \tilde{m}_2-1 \rangle}}^{\tilde{m}_2-1}}(2)\}| | [\overline{\Upsilon_{\mu_{\langle m, j+1 \rangle}}^{j+1}}(2) \overset{\rightsquigarrow}{\underset{B(\tilde{\mathcal{C}}_2)}{\longleftrightarrow}} \overline{\Psi_{\mu_{\langle m, \tilde{m}_2-1 \rangle}}^{\tilde{m}_2-1}}(2)] \succeq F_{p_{j, \tilde{m}_2}}. \quad (6.8)$$

But it is very simple to check that $F_p \succeq F_{\tilde{p}}$ when $1 \geq p \geq \tilde{p} > 0$, and we may therefore replace p_{j, \tilde{m}_2} by $p_{j,m}$ on the r.h.s. of (6.8):

$$|\{i: \overline{\Upsilon_{\mu_{\langle m, j+1 \rangle}, i}^j}(1) \overset{\rightsquigarrow}{\underset{B(\tilde{\mathcal{C}}_2)}{\longleftrightarrow}} \overline{\Psi_{\mu_{\langle m, \tilde{m}_2-1 \rangle}}^{\tilde{m}_2-1}}(2)\}| | [\overline{\Upsilon_{\mu_{\langle m, j+1 \rangle}}^{j+1}}(2) \overset{\rightsquigarrow}{\underset{B(\tilde{\mathcal{C}}_2)}{\longleftrightarrow}} \overline{\Psi_{\mu_{\langle m, \tilde{m}_2-1 \rangle}}^{\tilde{m}_2-1}}(2)] \succeq F_{p_{j,m}}. \quad (6.9)$$

Again we shall use (6.9) for all j down to \tilde{m}_3 .

Continuing for $s \leq v-1$ we extend the lower bounds for the probability of an active $\overline{\Upsilon_{\mu_{\langle m, \tilde{m}_{s+1} \rangle}}^{\tilde{m}_{s+1}}}(s)$ given the indices $\mu_{\langle m, \tilde{m}_{s+1} \rangle}$ yielded “active” hierarchical sets in the previous steps.

The construction at the final step $s = v$ is slightly different as remarked above, and we consider two cases: a) $\omega(B(m+1, \ell) > \omega(\tilde{\mathcal{C}}_v) + 1$; b) $\omega(B(m+1, \ell) = \omega(\tilde{\mathcal{C}}_v) + 1$.

In both cases we proceed as before as if $\tilde{m}_{s+1} = 0$, so that we use the analogue of (6.8) all the way down to $j = 0$. The only difference is that in case a) we again have a transfer operation, and we once more use Lemma 6.3, this time at scale 0, but in a space without bad layers and of vertical length at least L . In case b) we do not have the transfer operation, and the hierarchical set $\overline{\Upsilon^m}(v)$ stays on the last bad layer of $B_{\tilde{\mathcal{C}}_v}$.

In both cases, the final step to connect each final $\Upsilon_{\mu(m,0)}^0(v)$ to the matching $\Upsilon_{\mu(m,0)}^0$ has probability bounded from below by $p_G^\kappa p_B$.

Computing the probability. Verification of $(b_{m+1})'$. It is useful to establish a comparison with the following simple auxiliary scheme. Consider the following system of boxes: a unique $(m+1)$ -box (or box of scale $m+1$) contains J m -boxes, each of them containing J boxes of scale $m-1$, and so on down to scale 1: each 1-box contains J boxes of scale 0, thought as points.

Definition 6.5. *Checking procedure:*

- (a) Each 0-box is “good” with probability $p_G^{m\kappa} p_B^{m+1}$, all independently.
- (b) For each $k = 1, \dots, m-1$ a k -box is “good” if:
 - it contains at least one “good” $(k-1)$ -box;
 - it is “approved” at k -step, which happens with probability $p_k^{\kappa(m-k)}$ independently of everything else.
- (c) For $k = m, m+1$ a k -box is “good” if it contains at least one “good” $(k-1)$ -box.

With all “approvals” taken independently, and independent of the initial assignments (good/ not good), it is straightforward to see that for each $k = 0, \dots, m+1$, each k -box will be “good” with probability $p_{k,m+1}$.

Of course we could think of the previous procedure in two stages:

Stage 1

- (a) Each 0-box is “pre-good” with probability $p_G^{(m-1)\kappa} p_B^m$.
- (b) For each $k = 1, \dots, m-1$, a k -box is “pre-good” if:
 - it contains at least one “pre-good” $(k-1)$ -box;
 - it is “pre-approved” at k -step, which happens with probability $p_k^{\kappa(m-k-1)}$ independently of everything else.
- (c) For $k = m, m+1$ a k -box is “pre-good” if it contains at least one “pre-good” $(k-1)$ -box.

Stage 2 Each “pre-good” 0-box is “tested” again with probability p_G^κ ; if successful, it is declared “good”. In increasing order each k -box ($k = 1, \dots, m-1$) is “tested” again with probability p_k^κ , all “tests” being independently; if test is successful and if it contains at least one “good” $(k-1)$ -box, it is then declared “good”. For $k = m, m+1$, a k -box is declared “good” if it contains at least one “good” $(k-1)$ -box.

After taking into account the estimates obtained with the procedure based on the itinerary of the descending decomposition of the bad layer $B(m+1, \ell)$, we see that it is comparable (in the sense of stochastic order) with the previous “auxiliary scheme” with two stages: the first corresponds to the estimates provided by (6.5), (6.9) (at all steps $s = 1, \dots, v$), and the “testing at stage 2” comes from the “transfer” part, with the difference that the “test” with probability p_k^κ takes place only at $k = \tilde{m}_{s+1}$, for $s = 1, \dots, v$ along the itinerary (recall $\tilde{m}_{v+1} = 0$). At the scales which do not appear in the itinerary, the “test” is automatically successful with probability one.

Verification of $(b_{m+1})''$. The scheme used to define when a matching pair of $j+1$ -sets is chained, by taking at each step J separated matching j -sets then yields (conditional) independence (at each step), and allows to easily conclude $(b_{m+1})''$ from $(b_{m+1})'$.

To conclude the proof of Theorem 6.1, and therefore also of Theorem 4.11 in Section 4 (where p_G is taken close enough to 1), it remains essentially to show that by taking L large one can compare the numbers $p_{m-1,m}$ given by (6.1) with p_m defined immediately after (6.1) for all m . This will allow to conclude the induction step for (b_m) given by (4.36), summarized in the following:

Claim

Let $m \geq 2$. Assuming the validity of $(a_j), (b_j), (c_j), (d_j)$ for all $j \leq m-1$, and (a_m) , as explained immediately after (4.39) and (4.40), we can prove that (b_m) holds.

Taking into account what has been proven earlier in this section, it remains to verify that

$$8N(1 - p_{m-1,m})^{\rho \frac{c}{6} \frac{L}{N}} \leq q_m, \text{ for all } m \geq 2, \quad (6.10)$$

where N is given by (4.8), and $p_{m-1,m}, q_m, p_m$ are as in (6.1) and the line that follows it.

For this, and since L will be taken large it suffices to obtain

$$p_{m,m} \geq p_m \quad \forall m \geq 2. \quad (6.11)$$

Let

$$\Theta = \prod_{k=0}^{\infty} p_k > 0,$$

which is an increasing function of $p_G = p_0$, as also $\rho = \rho(p_G)$.

We recall the interpretation of $p_{m,m}$ given in Definition 6.5 (with $m+1$ now replaced by m), and proceed with a similar checking procedure, leaving the p_B^m -probability for the final step of the 0-boxes, i.e. with the trivial observation that if one has t (a fixed integer) independent Bernoulli random variables with probability of success given by $p\tilde{p}$, then the probability of no success is bounded from above by

$$(1 - \tilde{p})^{tp/2} + e^{-tI_p(p/2)}$$

where $I_p(x) = x \log(x/p) + (1-x) \log((1-x)/(1-p))$, for $x \in (0, 1)$, is the Cramér transform. (This follows at once from the decomposition of the Bernoulli essays into two independent ones, of probabilities \tilde{p} and p respectively, and Cramér Theorem for the second one.)

At all steps i from 0 to $m - 2$ each i -box is tested independently of anything else with probability $p_i^{\kappa(m-i-1)}$, and at the end the 0-box has to be approved with probability $\tilde{p} = p_B^m$.⁵ Using Cramér Theorem we can then estimate from above the probability that the m -box is not “good”, by splitting it into cases: (a) for each i , the number of tested $i - 1$ -boxes which are successful is not smaller than half of its expected number; (b) the event in (a) fails at some step i . Thus,

$$1 - p_{m,m} \leq (1 - p_B^m)^{4(J/2)^m \prod_{i=0}^{m-2} p_i^{\kappa(m-i-1)}} + \sum_{i=1}^{m-1} e^{-4(J/2)^{i+1} \prod_{j=2}^i p_{m-j}^{\kappa(j-1)} f(p_{m-i-1}^{i\kappa})} \quad (6.12)$$

with

$$f(p) = I_p(p/2) = (1 - \frac{p}{2}) \log(\frac{2-p}{1-p}) - \log 2 \quad (6.13)$$

It follows at once that L_0 large can be taken so that for all $m \geq 2$, and all $L \geq L_0(p_B, p_G)$,

$$(1 - p_B^m)^{4(J/2)^m \prod_{i=0}^{m-2} p_i^{\kappa(m-i-1)}} \leq (1 - p_B^m)^{4(J/2)^m (\Theta^\kappa)^{m-1}} \leq \frac{1}{2} q_m.$$

For the second term in (6.12), we split it into two pieces. For the piece corresponding to large values of i we use

$$\sum_{i=m/2}^{m-1} e^{-2(J/2)^{i+1} \prod_{j=2}^i p_{m-j}^{\kappa(j-1)} f(p_{m-i-1}^{i\kappa})} \leq \frac{m}{2} \exp \left\{ -2 \left(\frac{J}{2} \right)^{m/2} \Theta^{\kappa m} f(p_G^{\kappa(m-1)}) \right\}$$

which we can bound from above by $\frac{1}{4} q_m$ for all $m \geq 2$, provided $L \geq L'_0$ similarly as above. It remains to estimate

$$\sum_{i=1}^{m/2-1} e^{-4(J/2)^i \prod_{j=2}^i p_{m-j}^{\kappa(j-1)} f(p_{m-i-1}^{i\kappa})}.$$

Since we may assume (by taking L large) that $J\Theta^\kappa > 2$, this last term is bounded from above by

$$\frac{m}{2} - 1 \exp\{-2J\Theta^\kappa f(p_{m/2}^{\kappa m/2})\} \leq \frac{m}{2} \exp\{-4f(p_{m/2}^{\kappa m/2})\}.$$

To have this bounded from above by $\frac{1}{4} q_m$ we need $4f(p_{m/2}^{\kappa m/2}) > (m+1) \log q_0^{-1} + \log(2m)$, and a simple analysis of f given by (6.13) shows this is the case provided q_0 is chosen sufficiently small. Indeed, writing for convenience $q_0 = e^{-y}$, it remains to check

$$-\ln(1 - p_{m/2}^{\kappa m/2}) \geq \frac{1}{4} (\ln(4m) + (m+1)y) \quad (6.14)$$

⁵the m box and its $m - 1$ boxes are not tested, according to (6.1)

Assuming $\kappa m/2$ is an integer (small modification otherwise)

$$\begin{aligned}
1 - p_{m_2}^{\kappa m/2} &= 1 - \sum_{i=0}^{\kappa m/2} \binom{\kappa m/2}{i} (-1)^i e^{-i(m/2+1)y} \\
&= \sum_{i=1}^{\kappa m/2} \binom{\kappa m/2}{i} (-1)^i e^{-i(m/2+1)y} \\
&\leq \sum_{i=1}^{\kappa m/2} \binom{\kappa m/2}{i} e^{-i(m/2+1)y} \\
&\leq 2^{\kappa m/2} e^{-(m/2+1)y}
\end{aligned}$$

Thus for all such m

$$-\ln(1 - p_{m/2}^{\kappa m/2}) \geq -\frac{\kappa m}{2} \ln 2 + \left(\frac{m}{2} + 1\right)y \geq \frac{1}{4}(m+1)y,$$

provided $\frac{\kappa}{2} \ln 2 < y$, which holds for p_G sufficiently close to 1.

7. EXTENSION TO $p_G > p_c$

In order to extend the main result to all values $p_G > p_c$, several modifications of the scheme described in the previous sections are needed.

We start by choosing L large enough so that conditions (c_1) and (d_1) become satisfied. For this, the first thing is to enlarge the size of the 0-seed $Q^{(0)}$; it keeps the triangular shape but has K sites at its top line, with K large enough so that the probability of an infinite open oriented path (in the homogeneous p_G percolation model) starting from its adjacent sites from above has probability at least $1 - (1 - p^*)/4$. Adjusting c and increasing L one can check that the conditional probability of S^1 being s -passable given $Q^{(0)}$ is larger p^* , where the notion of passability at the level 1 includes new enlarged seeds on the top left and top right parts of S^1 .

Starting from this scale, the renormalization scheme repeats the previous one for the definitions of renormalized sites at scales $k \geq 2$, in particular the k -seeds, contain only three passable sites of scales $1 \leq j \leq k-1$.

At this point the only non-trivial modification involves the induction step for (b_m) done in Sections 5 and 6. From level m down to level 1 we follow the same procedure as before. The key change is in the definition of a pair of matching 1-sites being chained, and the corresponding probability estimate of such event. Assume that two sites S^1 and \hat{S}^1 form a matching pair with respect to a bad layer of mass m and level 1, and have s -dense kernel and respectively reverse \hat{c} -dense kernel. To concatenate open 0-sites in the cluster within $Ker(S^1)$ to some open 0-site in the reverse cluster within $Ker(\hat{S}^1)$, we will act differently from the case of large p_G since the density can now be arbitrary small. We look at the probability that out of the (order L) 0-sites lying below the bad layer and in the cluster within $Ker(S^1)$, at least $K' \geq K$ have disjoint open path crossing the bad layer. If this

occurs, one can see that with a probability compatible with the estimates in Section 6 at least one of these points will have an open 0 level path going to the top of \widehat{S}^1 . Due to the planarity, this establishes the desired connection.

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